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ELEMENTS

OF

PLANE TRIGONOMETRY,

SURVEYING AND NAVIGATION:

BY

WILLIAM SMYTH, A. M.,

PROFESSOR OF MATHEMATICS IN BOWDOIN COLLEGE.

BOSTON,
SANBORN, CARTER AND BAZIN.
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PREFACE.

In developing the subject of Plane Trigonometry, the object is to determine all the parts of a plane triangle from certain of these parts which are given.

We commence with the inquiry, how many of the parts must be known in order that the rest may be found. This being answered, the solution which first presents itself is the determination of the required parts by a construction of the triangle. In order to this we must have measures, both for the sides and angles. These being obtained, we proceed next to the construction, and solve all the different cases which may occur. In the performance of this work the student cannot fail to see the imperfection of the method, and the necessity, in order to accuracy, of a solution by arithmetical calculation. But here a difficulty presents itself, arising from the impracticability of combining, in the same numerical calculation, angles or the arcs by which they are measured, and straight lines; quantities, which having no common measure, cannot be compared with each other.

The problem not admitting of direct solution, some indirect method must be employed. On reflecting upon the means for accomplishing this object, it will occur that if a series of triangles can be found having angles of every possible magnitude, and the sides of which, or the ratio they bear to each other, are given, then any proposed triangle may be calculated from some one of these by means of a simple proportion. The inquiry now is, how can such a series of triangles be formed. Commencing with the right-angled triangle, as the most simple, we see that such a series may with facility be constructed in the quadrant of a circle; and with the advantage that the radius of the quadrant will be the common hypothenuse in all the triangles. But can the remaining sides of these triangles, or their ratio to the hypothenuse, be determined? This inquiry is next answered, and a series of rules developed, by which the desired object is accomplished. Calling the common hypothenuse radius, and the perpendiculars and bases of the triangles sines and cosines, respectively, of the arcs to which they correspond, we arrange next in a tabular form the values of these lines; and thus obtain our first idea of the nature and construction of the trigonometrical tables.

Proceeding next to the calculation of a triangle by the means now obtained, we soon reach a case in which the series of triangles already computed, cannot be employed; and a new series is required, in which the radius, instead of being the common hypothenuse of the triangles, shall be one of the sides. The next step in the development is to determine such a series, which is easily done. The new parts are called tangents and secants, and these being calculated are written down by the side of the corresponding sines and cosines in the same table. We have now the means for the complete solution of the right angled triangle. Availing ourselves of the advantage of logarithms in facilitating the calculations, we next arrange once for all, in a new set of tables, the logarithms of the sines, cosines, &c., already written in the tables first formed. The theory of the trigonometrical tables is now complete. But our results thus far apply only to right angled triangles. Can the method be extended so as to include the general case, that of the oblique angled triangle? By a simple demonstration it is next shown that, from the numbers already placed in the tables, a triangle may always be formed similar to any oblique angled triangle that may be proposed. A few simple rules for the different cases which may occur now complete the work. The problem proposed at the outset is completely solved.

We have now our Trigonometry. A simple application to a variety of problems in the determination of inaccessible heights and distances exhibits, at once, the vast practical utility of the branch of Mathematics thus developed.

Proceeding to the more general subject of Surveying, the same order of development is observed. After some preliminary definitions and description of instruments, the method of measuring the distance, in a variety of circumstances, between two points is first explained. Passing next to a simple field, the method of determining its bounding lines by means of the chain only, together with the plotting the work and determining its area, is then explained. For fields a little more complicated the want of some instrument for the ready determination of perpendicular lines is now felt. The Cross is therefore introduced, which gives us surveys with the chain and cross. Passing next to the use of some instrument for the measurement of angles, the want of which is now seen, we introduce the Surveyor's Compass, the most simple instrument of the kind, and the one in common use. We have thus surveys with the chain and compass.

With the means now in our hands, we connect with the bounding lines of fields the determination of the principal natural or artificial objects in their vicinity,—the first step towards the construction of a complete map of a section or portion of the earth's surface. From this we pass next to the survey of an estate, with its various inclosures, roads, buildings, &c.; a complete map of which is constructed.

The use of the compass in these surveys develops the inaccuracy of the instrument, especially in respect to the directive power of the needle. Means are, therefore, required for determining, in any place, the variation. In order to this, a true meridian line must be drawn; the method of doing which is now fully explained. The mode of surveying the Public Lands, by means of which the errors of the compass are either avoided or readily corrected, is also here introduced.

We are now prepared for surveys relative to various public works, such as Railroads, Canals, &c. The operation of leveling, the mode of determining the contour or form of the surface, the conventional signs by which the characteristics of the surface are determined, are next introduced; and various examples of surveys for the objects in question are given.

Passing to more extensive surveys, the Theodolite is now introduced; and the process of triangulation, and the manner in which all the details of the survey may be connected with the triangulation, are explained. Advancing next to surveys still more extended, all the leading operations required in surveys of the greatest extent, together with the mode of determining an arc of the meridian and ultimately the figure of the earth, are next developed.

The same general course is pursued in the Navigation.

In the Analytical Trigonometry the mode of obtaining, by the aid of Algebra, the various formulas required in the most extended use of the Trigonometrical lines, is explained.

The use of this treatise in Academies and High Schools seems to render this brief exposition of its plan desirable. The interest felt by the pupil in any object of inquiry, will depend, in some degree, upon the distinctness with which the object is placed before him, the orderly arrangement of the different steps of the process and their manifest relation to the attainment of the end desired. The Trigonometry, in particular, presents, it is believed, the advantages of a clearly defined and rigorous method. This should be kept in view by the teacher. At every point the pupil should see distinctly the progress he has made. An occasional review of the ground passed over, and of the whole at the end, with reference to the logical relations of the different steps in the discussion, and the manner in which each contributes to the final result, will awaken new interest in the study, and be in itself a most valuable source of mental discipline.

The Black Board will of course be employed in the demonstrations. In the Surveying it is specially recommended, that the examples be all drawn on the board, and the details filled in, as near as the eye can judge, in their just proportions and relations. A surprising degree of accuracy may be acquired in this process, so that the most extended survey may be presented upon the board with something of the precision of an actual plotting of the work. The exercise itself is a useful one, and imparts to the work of the recitation room, something of the interest and advantage of the actual operations in the field.

WM. SMYTH.

Bowd. Coll., July, 1855.

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BOOK I.

PLANE TRIGONOMETRY.

SECTION I.

PRELIMINARY REMARKS.

1. A figure bounded by three lines is called a *triangle.*

2. A triangle ABC (fig. 1), in which the bounding lines are straight lines, is called a *plane triangle.*

3. A plane triangle DEF (fig. 2), in which one of the sides DE forms with another EF a right angle, is called a *right angled triangle.* Any other triangle ABC (fig. 1), in which neither of the angles is a right angle, is called an *oblique angled triangle.*

4. In the right angled triangle DEF (fig. 2), the part DF, opposite the right angle, is called the *hypothenuse;* the other two parts DE, EF, are called the *sides;* sometimes also the *perpendicular and base.*

5. In a plane triangle there are six parts, viz. three angles and three sides. In order to determine the others, certain of these parts must be given.

6. A single part, it is evident, is not sufficient to determine the rest. Let us begin with two.

And first, let the lines *a* and *b* (fig. 3), be two of the sides of a triangle. The lines *a* and *b*, it is evident, may make with each other any angle whatever. They may, therefore, be sides in an indefinite number of triangles, ABC, ABC′, &c. The other parts of a triangle cannot, therefore, be determined, when two of the sides are the only parts which are given.

Again, let a side AB, and an angle A (fig. 4), be two given parts of a triangle. The other parts cannot be determined by these; for the side AB and the angle A may belong to an indefinite number of triangles ABC, ABC′, &c.

Moreover, if two of the angles, or, which is the same thing, if all the angles are given, the remaining parts will still be undetermined; for in the triangle ABC (fig. 5) if the lines B′C′, B″C″, &c. are drawn parallel to BC, an idefinite number of triangles ABC, AB′C′, &c. may be formed, in each of which the angles will be respectively equal, while the sides are of different magnitudes.

In order, then, to determine the remaining parts of a triangle, *three, at least, of the parts must be given, one of which must be a side.*

7. It will be seen in what follows that the remaining parts of a triangle may always be determined, when three of the parts are given, provided that one of these is a side.

That branch of science which teaches how to find the remaining parts of a plane triangle from certain parts which are given, is called *Plane Trigonometry.*

SECTION II.

SOLUTION OF TRIANGLES BY GEOMETRICAL CONSTRUCTION.

8. The method of solving a triangle from certain of the parts which are given that first suggests itself, is, to construct the tri-

angle by means of the given parts; and then to measure the required parts, in the use of the instruments by which the triangle is constructed.

9. Among the parts of a triangle there are two kinds of quantities, viz. sides and angles, for each of which measures must be found.

10. The sides of a plane triangle being right lines, are expressed in the usual linear measures, as feet, yards, &c. To represent these measures, for the purpose of calculating a triangle geometrically, any line AB (fig. 6) may be divided into a convenient number of equal parts; one of these being considered as unity may be further divided into equal portions for fractional parts of unity. A line divided in this manner is called a *scale of equal parts.*

Thus in figure 6, the line AB is divided into eleven equal portions; of these the extreme one on the left is divided into ten equal parts, the remainder being numbered 1, 2, 3, &c. The principal divisions of this scale may be considered as feet, miles, &c.; then the subdivisions will be tenths of a foot, mile, &c. The principal divisions may also be regarded as ten feet, ten miles, &c.; in this case the subdivisons will represent feet, miles, &c.

A convenient construction for a scale of equal parts is represented in figure 7. It consists, when intended for a decimal scale, of eleven lines drawn parallel to each other at equal distances, and divided into convenient portions by perpendicular lines. In the extreme division on the left, the upper and lower lines are divided into ten equal parts, and the subdividing points are connected by *diagonal* lines; that is, by lines drawn from the first point on the lower line to the second on the upper, &c. Then, by similar triangles, $ac : ab :: cd :: bo$; that is, cd is one tenth of the subdivisions, or one hundredth of the primary divisions of the scale. In like manner, the corresponding portion on the second parallel line may be shown to be two hundredths, and that on the third parallel line three hundreths of the primary divisions of the scale.

Let it be required, for example, to take off the number 255 from this scale. Considering the subdivisions as containing ten each, we extend the dividers from figure 2 of the principal divisions to 5 of the subdivisions for 250; then by opening the dividers to the corresponding extent on the fifth of the parallel lines, we shall obtain the length required.

11. In order to estimate the magnitude of the angles, it would seem more natural to refer them to the right angle, as the unit of measure; since the quantities to be measured would in this case, be the same in kind with the quantity assumed as their measure. But it is found more convenient in practice to measure angles by arcs of circles. In the circle AC (fig. 8), in whatever ratio the angle DBC at the centre increases or diminishes, the arc CD, on which it stands, increases or diminishes in the same ratio (Geom.); the arc CD may, therefore, be assumed as the measure of this angle. In like manner any other concentric arc EF, intercepted by the sides of the angle DBC, may be used as its'measure. The arc CD, indeed, considered as a magnitude of a certain length, is manifestly greater than the arc EF. In the measure of angles, however, it is not the absolute value of the arcs, which we regard, but merely their ratio to an entire circumference. Since, then, the arc EF is to the circumference EG, as the arc CD to the circumference CA; considered as parts of the entire circumferences, to which they respectively belong, the arcs CD and EF are equal in quantity. We may take, then, as the measure of an angle *the arc of a circle, having its centre in the vertex of the angle, and intercepted by the lines which form its sides.*

In the measure of an angle it is obviously immaterial what may be the size of the circle employed, provided it cut the sides of the angle. In comparing different angles, however, the arcs, on which they are measured, should be described with equal radii.

12. The entire circumference of every circle, whether great or small, is divided into 360 equal parts called degrees; each degree into 60 equal parts called minutes; each minute into 60 equal parts

called seconds; each second into 60 equal parts called thirds, &c. The character used to denote degrees is °, placed at the right hand of the unit figure and a little above it; that for minutes is ′, that for seconds is ″, that for thirds is ‴; so that 40° 30′ 25″ 10‴ signifies 40 degrees, 30 minutes, 25 seconds, and 10 thirds. This division is entirely arbitrary, and any other may be assumed at pleasure. We may, for instance, after the manner of the French, divide the circle into 400 equal parts for degrees; and each degree into 100 equal parts for minutes; with further subdivisions according to the principles of the decimal progression.

13. We have seen (art. 11) that the measure of an angle DBC (fig. 8) is the arc CD of a circle CA; but the chord CD, being the distance in a right line between the extremities of the arc CD, will determine the magnitude of this arc. If, then, the several chords for each degree, minute, &c., contained in the arc AB (fig. 9) equal to a quadrant or fourth part of a circle, be transferred to the line BD, the line BD, denominated in this case a *line of chords*, may with convenience be employed in the measurement of angles.

Let it be required, for example, to ascertain the magnitude of the angle ABC (fig. 10) by the line of chords. With the extent from 0 to 60 on the line of chords, we describe, from B as a centre, an arc cutting in the points *m* and *n* the two lines which include the angle; then, since the chord of 60° is equal to radius (Geom.) this arc will have the same radius with that to which the given scale of chords belongs; consequently, the distance between the points of intersection, *m* and *n*, applied to the line of chords, will give the measure of the angle in degrees.

Conversely, let it be required to make an angle ABC of a given magnitude, 30° for example. Draw a line BC; then from the point B, with a radius B*n* equal to the chord of 60° on the line of chords, describe an arc *nm*; the extent from B to 30° on the line of chords, applied from *n* on the arc *nm*, will give the point through which, if the line BA be drawn, the angle ABC will be formed of the required magnitude.

If the angle required is greater than 90°, we first, set off on *nm* with the chord of 90, an arc of 90°; then setting off from this the excess of the given angle above 90°, we shall have the point through which to draw the remaining side of the angle.

14. We are now prepared for the solution of triangles, when three, the required number of parts, are given. The different Problems which may occur are 1°, Given the three sides; 2°, two sides and an angle; 3°, one side and two angles; to find the remaining parts.

PROBLEM I.

Given *three sides* of a triangle, to find the remaining parts.

Ex. 1. Let the sides of the triangle be 39, 45, and 37 feet respectively.

By means of the scale of equal parts we lay down one of the sides AB, for example (fig. 1), equal 45. From the same scale we then take in the dividers one of the remaining sides 39, and with one foot in the point A, as a centre, we describe an arc. Taking next in the dividers the remaining side 37, we describe from the point B, as a centre, another arc cutting the former in the point C. We then draw AC and BC, and the triangle ABC will be the triangle required. The angles may then be measured upon the line of chords in the manner explained above.

Ex. 2. Given the three sides 53.5, 82.08, and 46.7; to find the angles.

Ex. 3. Given the three sides 67, 75.9, and 121.05; to find the angles.

PROBLEM II.

Given *two sides and an angle*, to find the remaining parts.

The problem admits of several cases.

1. Two sides and the included angle being given.

Ex. 1. Two sides of a triangle are 60, and 100 rods respectively, and the angle included is 45° 30′. What are the remaining parts?

We first draw one of the given sides BC, for example, (fig. 11) equal to 60. At the point B, one of the extremities of BC, we make next an angle ABC equal to 45° 30′, the given angle; through this we then draw the other given side BA equal to 100; and connecting the points A and C, ABC will be the triangle required. The required parts may then be measured; the side AC on the scale of equal parts, and the angles A and C upon the line of chords.

Ex. 2. Given two of the sides of a triangle equal to 354.7, and 563.03 feet respectively, and the angle contained between them 53° 45′, to find the remaining parts.

Ex. 3. Two sides of a triangle are 75.3, and 89.75 yards respectively, and the included angle is 97°. What are the remaining parts?

2 Two sides and an angle opposite the longer given side.

Ex. 1. Let the given sides be 57.5, and 103 feet respectively, and the angle opposite 103, the longer side, be 40°; required the remaining parts.

We draw the shorter given side AB (fig. 12) equal to 57.5, and at one extremity A of this side we make the angle BAC equal to 40°, through which we draw an indefinite line AC; then, with an extent equal to 103, the other given side, we describe from B, as a centre, an arc cutting the indefinite line AC at C, and join BC. ABC will be the triangle required; the remaining parts of which may be measured as above.

Ex. 2. Given one side of a triangle equal to 93 feet, another side equal to 179 feet, and the angle opposite this last equal to 63° 30′; to find the remaining parts.

3. Two sides and an angle opposite the shorter given side.

Ex. 1. Given in the triangle ABC (fig. 13) the side AB 72 feet, the side AC 41,5 feet, and the angle B 30°; required the remaining parts.

In this example, in which the side opposite the given angle is less than the other given side, the solution is *ambiguous*; for if from the point A, with a radius equal to AC, we describe the arc CC′ cutting the line BC in C′, and draw AC′, a second triangle ABC′ will be formed which will also contain the three given parts. To determine, then, which of these triangles is intended in the question proposed, we must know, in addition to the given parts, whether the angle C be acute or obtuse. With respect to this case, we may remark in passing, that the angles AC′B, ACB are together equal to two right angles. Indeed the angles AC′B, AC′C are together equal to two right angles; but by construction the triangle AC′C is isosceles; the angle ACB is therefore equal to AC′C; whence AC′B, ACB are together equal to two right angles. The angles ACB, AC′B are, therefore, supplements, one of the other; since we understand by the *supplement of an angle that which must be added to this angle in order to make two right angles.*

Ex. 2. Given two sides of a triangle equal to 530, and 253 respectively, and the angle opposite the shorter side equal to 28°; required the remaining parts.

PROBLEM III.

Given *one side and two angles*, to find the remaining parts.

Ex. 1. Given one side of a triangle equal to 120 yards, and the adjacent angles equal to 50°, and 63° respectively, to find the remaining parts.

The solution of this problem is left as an exercise for the learner. It will be observed, however, that if the given angles are not both adjacent to the given side, we must subtract their sum from 180°, which will give the other adjacent angle.

Ex. 2. Given the side AB of a triangle (fig. 14), 40 feet, the angle ABC 52°, and the angle BAC 70°, to find the remaining sides.

Ex. 3. Given one side of a triangle 600 yards, the angle opposite 65°, one of the adjacent angles 40° 30′; required the remaining parts.

15. It is now evident, that we may always construct a triangle, when three of its parts are given, provided that one of them be a side. If, however, the given parts are two of the sides and an angle opposite one of them, and the side opposite the given angle is less than the other given side, in order to determine the triangle we must know whether the angle sought be acute or obtuse.

16. From the preceding examples it is easy to see, that the solution, obtained by geometrical construction, will not, on account of the imperfection of instruments, be precisely the truth, but only a rough approximation to it. It is then required to ascertain methods of calculating triangles by numerical operations; since by means of these the calculation may be carried to any degree of exactness at pleasure.

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SECTION III.

OF THE SOLUTION OF TRIANGLES BY ARITHMETICAL CALCULATION.

17. In the solution of a triangle, it is obvious, that the magnitude of the angles will always come into consideration. Thus in the triangle ABC (fig. 1), let it be required to determine the third side AC, the angle at B and the sides AB, BC being given. Since the value of AC evidently depends in part upon the magnitude of the angle at B, it is manifest, that the magnitude of this angle must be made to enter into the computation. But the angle at B is measured on the arc of a circle, while the sides are right lines; by consequence the parts, which are necessarily the elements of the calculation, are different in kind and do not admit of comparison with each other. In the effort, then, by means of numerical operations, to determine the different parts of a triangle, we are embarrassed at the outset by the difficulty of introducing into the calculation the magnitude of the angles.

18. To show, in a particular case, the manner in which this

difficulty is avoided, we suppose that the triangles ABC, *abc* (fig. 15) are similar, and that the sides *ab*, *bc*, *ac*, of the latter are known. If, then, in the triangle ABC a side BC be given, it is evident from the nature of similar triangles, that the remaining parts of this last may be determined from those of the other by means of a simple proportion.

Let it be supposed, for further illustration, that we have determined, by geometrical construction, the ten triangles in Plate 11; the first having the angle at B 10°, the second 15°, the third 20°, and so on; and, for the sake of simplicity, let these triangles each be right angled at C. If it now be proposed to solve the triangle *abc* (fig. 16), right angled at *c*, the side *bc* 73 feet, and the angle *b* 40°, it is evident, on inspecting the series of computed triangles, that the triangle No. 7 is similar to the triangle proposed; since these triangles are both right angled, and one of the acute angles in the one is equal to an acute angle in the other. By means, then, of the computed triangle in the series, the remaining sides in the proposed triangle may be determined.

Thus to find *ac*, we have the proportion, as the side BC is to AC in the computed triangle, so is the side *bc* to *ac* in the triangle proposed; that is,

$$90 : 72 :: 73 ; 58.4 \text{ the side } ac \text{ required;}$$

and to find *ab* we have

$$BC : AB :: bc : ab; \text{ or, } 90 : 115 :: 73 ; 93.3 \text{ the side } ab.$$

In like manner, we may determine any right angled triangle whatever, provided that one of its acute angles be equal to an acute angle in one of the triangles of the series. We may form, moreover, a more extended series of right angled triangles, by making in the first triangle an angle of one degree; in the second an angle of two degrees, in the third of three, and so on, to eighty-nine degrees inclusive. In this case there will always be found, among the triangles of the series, one similar to any right angled triangle that may be proposed; provided that one of its acute angles be an entire number of degrees.

In like manner, it is evident, we may so extend the series of computed triangles, that there will always be found, among the number, a triangle similar to the one which shall be proposed for calculation; by consequence, as in the preceding example, the parts of this last may be determined from the former by means of a simple proportion.

19. Thus far we have supposed the series of triangles actually constructed, and arranged as in Plate II, for the calculation of other similar triangles. Such an arrangement, however, would obviously be inconvenient, especially as the series is more extended. To avoid this inconvenience, we may write in a table the different parts in the several triangles of the series, expressing those, which correspond in each, by the same general term, and arranging them in a uniform order.

20. To form a table, for example, of the triangles, which we have determined Plate II, we write down in order in the first column, as below, the angles at B in the several triangles; in the second the hypothenuses, in the third the bases, and in the fourth the perpendiculars; recollecting that by the term base we understand the side adjacent the angle at B in each of the triangles, and by the term perpendicular the side opposite this angle.

Deg.	Hyp.	Base	Perp.
10	242	237	43
15	127	123	33
20	121	113	42
25	139	125	59
30	132	114	65
35	125	104	70
40	115	90	72
45	103	72	72
50	93	61	69
55	132	73	109

21. To apply the table, let it be supposed, that we know in the right angled triangle ABC (fig. 17) the angle at B 35°, and the side BC 200 feet. In order to solve this triangle, we look along the column of degrees for 35; against this we find 104 the side of the tabular triangle corresponding to the given side in the triangle proposed. We then have the following proportion,

As 104, the base of the tabular triangle
: 125, the hypothenuse
:: BC 200 feet, the base of the proposed triangle
: BA 240.3 feet, the hypothenuse.

And, to find the perpendicular

base		perp.		BC		AC
104	:	70	::	200	:	134.6.

We thus find for the hypothenuse and perpendicular of the proposed triangle 240.3 feet, and 134.6 feet respectively.

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SECTION IV.

METHOD OF FORMING A SERIES OF TRIANGLES HAVING ANGLES OF EVERY POSSIBLE MAGNITUDE.

22. Having seen the use, which may be made of a series of triangles containing angles of every possible magnitude, and the sides of which are calculated, we next inquire into the method of constructing such a series. For the sake of simplicity let the triangles to be determined be right angled. It is evident (fig. 18), that such a series of triangles may be constructed in the quadrant of a circle. For if from each point in the arc AB, we let fall the perpendiculars ED, E′D′, E″D″, &c. upon the radius AC, and draw the radii EC, E′C, E″C, &c. the triangles EDC, E′D′C, E″D″C, &c. thus formed are right angled at D, D′, D″, &c.; and the angles ECD, E′CD′, E″CD″, &c. will be successively of every possible

magnitude. And since any two right angled triangles are similar when an acute angle in one is equal to one of the acute angles in the other, there cannot be proposed a right angled triangle, which will not be similar to some one of the triangles constructed in this manner.

23. To form a table from the triangles of this series, we employ, as before, convenient terms to express their sides. The hypothenuses CE, CE′, CE″, &c. being all radii of the same circle, may be designated by the term *radius*. The perpendiculars ED, E′D′, E″D″, &c. manifestly vary with the angles ECD, EC′D′, E″CD″, &c., to which they are opposite, or which is the same thing, with the arcs AE, AE′, AE″ &c., by which these angles are measured. On account of this connection we designate the perpendiculars ED, E′D′, E″D″, &c., by the term *sine*, derived from the Latin term *sinus*, signifying a bay or curve. We then define the sine of an arc, *the perpendicular let fall from one extremity of this arc upon the radius, which passes through the other extremity.*

In order to decide upon a term to express the remaining side, we now remark, that two *arcs are said to be complements the one of the other, when the sum, or difference of these arcs, is equal to the fourth part of the circumference of a circle.* From the definition already given of a sine, the line EF, considered with reference to the arc EB, is the sine of that arc; but EB is the complement of AE; EF, therefore, is the sine of the complement of AE, and may for the sake of brevity be called its *cosine.* But the side CD of the triangle CDE is equal to EF, since these lines are parallel to each other, and are included between parallels; CD may therefore be considered the cosine of AE, and CD′ of AE′, &c.; whence the remaining side of the triangles CDE, CD′E′, &c. may be designated by the term *cosine*; and we say, that *the cosine of an arc is the sine of the complement of this arc;* and *is equal to that part of the radius comprehended between the centre and the foot of the sine.*

24. From what has been said, it will now be recollected, that in constructing, in the quadrant of a circle, a series of right angled triangles having angles of every possible mignitude, the radius of the quadrant forms the common hypothenuse in the several triangles, and that the remaining sides are respectively the sines and cosines of the acute angle which has its vertex at the centre.

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SECTION V.

METHOD OF CALCULATING THE SIDES IN THE SERIES OF TRIANGLES, CONSTRUCTED IN THE QUADRANT OF A CIRCLE.

25. We have now ascertained a convenient method of constructing a series of triangles, in which the angles are of every possible magnitude. But the triangles of this series being determined by construction, no higher degree of accuracy is attained in the use of them. Let us next see if rules can be found by which they can be determined by arithmetical calculation.

And first, it is evident, that it will be sufficient merely to know the value of the sides as compared with each other. We may, therefore, adopt the common hypothenuse, or radius, as the unit; and the question will be, to determine the values of the sines and cosines in decimal parts of this unit; or, which is the same thing, we may consider the radius as divided into 100 000, or any other convenient number of equal parts, and then determine the number of these parts in each of the sines and cosines.

26. It is evident from inspection (fig. 18), that the sine of 90° is equal to radius. And since (fig. 19) the side of an inscribed hexagon is equal to radius (Geom.); if through the centre E of the arc BC we draw the radius EF, the chord BC will be bisected at G, and BG the sine of the arc BE will be equal to half of the radius; but the arc BE is equal to one third of a quadrant, or thirty

degrees; there are, then, two of the sines in a quadrant, the values of which we may know at the outset; viz. the sine of 90° equal to radius, and the sine of thirty degrees equal to one half of radius.

27. Beginning with the sine BG (fig. 19,) we find immediately its cosine GF; for in the triangle BGF, which contains these sides, and the hypothenuse of which is equal to radius or unity, we have

$$BF^2 = BG^2 + GF^2; \text{ whence } GF = (BF^2 - BG^2)^{\frac{1}{2}};$$

that is, $\cos EB = (R^2 - \sin EB^2)^{\frac{1}{2}}$.

To find, therefore, the cosine of an arc, when the sine is given; *from the square of the radius subtract the square of the sine, the square root of the remainder will be the cosine.*

28. From what has been said, the relative value of the sides in one of the triangles of the series which we wish to calculate may now be considered as known; we are next led to inquire into the use which may be made of these in the calculation of the remaining triangles of the series.

29. If the arc AE (fig. 20) be divided into two equal parts by the radius CD, the chord AE will also be divided by it into two equal parts; and EG will be the sine of the arc DE one half of AE. The triangle AEF, right angled at F, gives $AE^2 = AF^2 + EF^2$; but $AF = AC - FC = R - \cos AE$, and $EF = \sin AE$; squaring these values of AF and EF and substituting the squares in the expression for AE^2 we have

$$AE^2 = \sin AE^2 + R^2 - 2R \times \cos AE + \cos AE^2;$$

but $EF^2 + FC^2 = CE^2$; or $\sin AE^2 + \cos AE^2 = R^2$;

hence $AE^2 = 2R^2 - 2R \times \cos AE$;

whence, extracting the root and dividing by 2,

$$EG = \tfrac{1}{2}(2R^2 - 2R \times \cos AE)^{\frac{1}{2}}.$$

But EG is the sine of one half the arc AE; whence to find the sine of half any arc less than a quadrant; *from twice the square of the radius subtract twice the radius multiplied by the cosine* *of the given arc ; one half the square root of the remainder will be the sine of half this arc.*

30. Let it now be supposed that we wish to calculate a series of triangles, in which there will always be found a triangle having for one of its acute angles any possible number of minutes, or in other words, that we wish to calculate a series of triangles for every minute in the quadrant. By art. 25 we have radius equal to 1.00000, or, omitting the separatrix, to 100000; the sine of 30° will then be equal (art. 26) to 50000, and its cosine (art. 27) to 86603. By the rule last obtained we may now calculate the sine of 15°, one half of 30°, the sine of 7° 30′, one half of 15°, &c. Proceeding in this manner we obtain at the eleventh division, as below, the sine of $52'' \; 44''' \; 3'''' \; 45'''''$ equal to 00025, or calling the radius 1000000, to 000255.

	Sine	Cos.
30°	50000	86603
15°	——	——
7°30′	——	——
3°45′	06540	99786
——	——	——
——	——	——
——	——	——
$0° \, 14' \, 3'' \, 45'''$	00409	99999
——	——	——
——	——	——
$0° \, 0' \, 52'' \, 44''' \, 3'''' \, 45'''''$	00025	——

We next inquire into the manner, in which the sine of one minute may be obtained from this last.

31. It is evident (fig. 21) that if the arcs A*b*, A*c* are very small, they will not differ materially from right lines; the triangles A *b d*, A *c e* may then be considered similar, and A*b* is to A*c*, as *bd* to *ce*. In very small arcs, therefore, *the ratio of the arcs is so nearly equal to the ratio of their sines, that the one may be taken for the other without sensible error.* The error, indeed, in supposing that arcs less than one minute are proportional to their sines will not affect the first ten places of decimals.

To find the sine of one minute from the sine of $52''\,44'''\,3''''\,45'''''$, we have, then, the following proportion,

$52''\,44'''\,3''''\,45''''' : 60'' :: 000255 : 00029$ the sine of one minute.

32. The sine and cosine of one minute may now be considered as known, and by consequence all the parts in the first triangle of the series we wish to calculate. We proceed to the investigation of rules, by which the remaining triangles of the series may be computed from this first.

Let it be supposed, that the sines of the two arcs AB, AC (fig. 22) are known; it is proposed to find the sines of the sum and difference of these arcs. To construct the figure, make the arc AD equal to the arc AC; draw the chord CD and the radius LA; the radius LA will divide this chord into two equal parts at the point I (Geom.); from the points C, A, I, and D, let fall upon BL the perpendiculars CK, AG, IH and DF; lastly, at the points I and D, draw IM and DN, parallel to BL. Then CK, the sine of BC, the sum of the two arcs AB, AC, is equal to KM + MC; but KM is equal to HI; CK is therefore equal to HI + MC; and the sum of the values of HI and MC will be equal to the sine of the sum of the two arcs AB, AC.

To find the value of HI, we have

$$\mathrm{LA} : \mathrm{LI} :: \mathrm{AG} : \mathrm{HI},$$

or putting for these lines what they severally denote

$$\mathrm{R} : \cos \mathrm{AC} :: \sin \mathrm{AB} : \mathrm{HI};$$

hence
$$\mathrm{HI} = \frac{\sin \mathrm{AB} \times \cos \mathrm{AC}}{\mathrm{R}};$$

and since the triangles LAG, CIM are similar, to find MC we have

$$\mathrm{LA} : \mathrm{LG} :: \mathrm{CI} : \mathrm{MC}; \text{ or, } \mathrm{R} : \cos \mathrm{AB} :: \sin \mathrm{AC} ; \mathrm{MC};$$

hence
$$\mathrm{MC} = \frac{\sin \mathrm{AC} \times \cos \mathrm{AB}}{\mathrm{R}}.$$

Adding together these two expressions, we have

$$\sin(\mathrm{AB} + \mathrm{AC}) = \frac{\sin \mathrm{AB} \times \cos \mathrm{AC} + \sin \mathrm{AC} \times \cos \mathrm{AB}}{\mathrm{R}}.$$

Then, since CD is divided into two equal parts at the point I, CN is also divided into two equal parts at the point M, and DF the sine of BD, the difference of the two arcs AB, AC is equal to KN; that is, to KM — MC, or which is the same thing, to HI — MC. To find, then, the sine of the difference of the two arcs AB, AC, we subtract the one from the other the values of the lines HI, MC already obtained, and

$$\sin(AB - AC) = \frac{\sin AB \times \cos AC - \sin AC \times \cos AB}{R}.$$

Having calculated, therefore, the cosines of any two arcs, the sines of which are given, in order to find the sine of the sum and of the difference of these arcs; *multiply the sine of the first by the cosine of the second, and the sine of the second by the cosine of the first; the sum of the two products divided by radius will be the sine of the sum of the two arcs; and the difference of the same products divided by radius will be the sine of the difference of these arcs.*

33. The sine of the sum and of the difference of any two arcs being calculated by the rule now obtained, the cosines may be calculated by art. 27. When, however, the sines of any two arcs AB, AC (fig. 22) are given, the cosine of the sum, and of the difference of these arcs, may be calculated in a more convenient manner.

Since DC (fig. 22) is divided into two equal parts in I, FK will be divided into two equal parts in H; and LK, the cosine of BC, the sum of the arcs AB, AC, will be equal to LH — HK, or to LH — IM. To find LH, we have

$$LA : LI :: LG : LH;$$

that is, $$R : \cos AC :: \cos AB : LH;$$

whence $$LH = \frac{\cos AB \times \cos AC}{R}.$$

To find IM, the similar triangles LAG, CIM give

$$LA : AG :: CI : IM; \quad \text{or } R : \sin AB :: \sin AC : IM;$$

whence $$IM = \frac{\sin AB \times \sin AC}{R}.$$

Subtracting this last expression from the former, we have

$$\cos(\mathrm{AB}+\mathrm{AC})=\frac{\cos\mathrm{AB}\times\cos\mathrm{AC}-\sin\mathrm{AB}\times\sin\mathrm{AC}}{\mathrm{R}}.$$

Moreover LF, the cosine of BD, the difference of the arcs AB, AC, is equal to LH + HF; or since HF is equal to IM, to LH + IM. To find, therefore, the cosine of the difference of the arcs AB, AC, we add together the values of LH and IM already found, and

$$\cos(\mathrm{AB}-\mathrm{AC})=\frac{\cos\mathrm{AB}\times\cos\mathrm{AC}+\sin\mathrm{AB}\times\sin\mathrm{AC}}{\mathrm{R}}.$$

To find, therefore, the cosine of the sum, and of the difference of any two arcs, whose sines are given; having calculated the cosines of these arcs, *take the product of the cosines, and also of the sines; the difference of these two products divided by radius will be the cosine of the sum of the two arcs; and the sum of the same products divided by radius will be the cosine of their difference.*

34. From the equation (art. 32)

$$\sin(\mathrm{AB}+\mathrm{AC})=\frac{\sin\mathrm{AB}\times\cos\mathrm{AC}+\sin\mathrm{AC}\times\cos\mathrm{AB}}{\mathrm{R}},$$

we may find the sine of twice any arc, whose sine and cosine are given; for as the arcs AB, AC, may be of any dimensions, they may be considered equal; substituting, then, AB for its equal AC throughout, we have

$$\sin 2\,\mathrm{AB}=\frac{2\sin\mathrm{AB}\times\cos\mathrm{AB}}{\mathrm{R}}.$$

To find, then, the sine of twice an arc, whose sine and cosine are given; *multiply twice the sine of this arc by its cosine, and divide the product by radius.*

35. From the equation (art. 33),

$$\cos(\mathrm{AB}+\mathrm{AC})=\frac{\cos\mathrm{AB}\times\cos\mathrm{AC}-\sin\mathrm{AB}\times\sin\mathrm{AC}}{\mathrm{R}},$$

we may likewise find the cosine of twice any arc, when its sine and cosine are known; for supposing the two arcs equal, as before, and substituting AB for its equal AC throughout, we have

$$\cos 2\,AB = \frac{\cos AB^2 - \sin AB^2}{R}.$$

Hence, when the sine and cosine of an arc are known, to find the cosine of twice that arc; *subtract the square of the sine of the arc from the square of its cosine, and divide the remainder by radius.*

36. We have seen (art. 31) the manner in which the sine of one minute may be calculated; consequently, all the parts in the first triangle of the series, which we wish to compute, may be considered as known. By the rules (art. 34, and 35) for the sine and cosine of twice any arc, we may now obtain from this first triangle the sine and cosine of two minutes, or the second triangle; and by the repeated application of the rules, art. 32, and 33, the remaining triangles of the series may be calculated.

Ex. From the data given below it is required to calculate the sine and cosine of 10°, 11°, &c, extending the calculation to six places of figures; that is, supposing the radius to be divided into 1000000 equal parts.

	Sin.	*Cos.*
1°	017452	999848
5°	087156	996195
10°	———	———
11°	———	———
12°	———	———
13°	224951	974370
14°	———	———
15°	———	———

37. We now perceive the advantage of constructing the series of triangles, we wish to compute, in the quadrant of a circle; since, if the radius of the quadrant be considered as unity, the value of one of the sides in a triangle of the series will be known; and from this, radius being common to all the triangles, the remaining sides in each of the triangles of the series may be determined.

38. Having calculated a series of triangles by the rules now obtained, the different parts of this series may be arranged in a table as in art. 20. The radius, however, which is common to all the triangles may be omitted, or written once only at the head of the table.

The following table, containing the sines and cosines for each degree from 26° to 45° inclusive, and calculated to five places, is sufficient for our present purpose.

Deg.	Sine	Cosine	Deg.	Sine	Cosine
26	43837	89879	36	58779	80902
27	45399	89101	37	60182	79864
28	46947	88295	38	61566	78801
29	48481	87462	39	62932	77715
30	50000	86603	40	64279	76604
31	51504	85717	41	65606	75471
32	52992	84805	42	66913	74314
33	54464	83867	43	68200	73135
34	55919	82904	44	69466	71934
35	57358	81915	45	70711	70711

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SECTION VI.

SOLUTION OF TRIANGLES BY THE TABLE OF SINES AND COSINES.

39. To apply the table of sines and cosines, let there be given for solution the right angled triangle ABC (fig. 23), in which the angle at B is 30°, and the hypothenuse AB three yards.

Since we have regarded the radius of the table as unity without reference to any particular denomination of measure, it may manifestly, in all cases, be regarded as a unit of that species, in which the sides of a proposed triangle are expressed. With an extent then equal to one yard, which in the present instance will be the radius of the table, describe from B the arc DF; from D let fall DE perpendicular to BC; then will DE be the sine, and BE the cosine of the angle at B, and DBE will be the triangle of the table similar to the proposed triangle ABC. To find the side AC, we then have BD : DE : : BA : AC; or R : sin B : : BA : AC

a proportion which may be enunciated as follows; *radius is to the sine of one of the acute angles in a right angled triangle, as the hypothenuse to the side opposite.*

Again, to find BC we have

BD : BE : : BA : BC; or, R : cos B : : BA : BC,

a proportion which may be enunciated as follows; *radius is to the cosine of one of the acute angles, as the hypothenuse to the side adjacent.*

40. We pass next to some examples.

Ex. 1. In a right angled triangle CDE (fig. 24), given the hypothenuse CD, 250 feet, and the angle D, 31°, to find the remaining parts.

1. To find the perpendicular CE, we have

R : sin D : : CD : CE;

but from the preceding table we find sin 31° = 51504, radius being 100000; hence

100000 : 51504 : : 250 ft. : 128.76 ft.

the perpendicular required.

2. To find the base we have

R : cos D : : DC : DE;

or 100000 : 85717 : : 250 ft. : 214.29 ft,

the base required.

Ex. 2. In a right angled triangle CDE (fig. 24), given the hypothenuse 325 yards, and the perpendicular 200.09 yards; to find the remaining parts.

To find the angle D, we have

CD : CE : : R : sin D;

or 325 : 200.09 : : 100000 : 61566.

From inspection of the table we find that 61566 in the column of sines corresponds to 38°. The angle D is, therefore, 38°. The angle D being now known, the base will readily be found equal to 256.1 feet.

Ex. 3. Given the hypothenuse 329, and the base 244.493; required the remaining parts.

Ans. The angle adjacent to the base is 42°, and the perpendicular is 220.144.

41. From the preceding example, it is evident, that we may determine, by means of a table of sines and cosines, one of these three things, when two of them are given, viz. *the hypothenuse, a side and one of the acute angles.*

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SECTION VII.

SECOND METHOD OF CONSTRUCTING AND CALCULATING A SERIES OF TRIANGLES, HAVING ANGLES OF EVERY POSSIBLE MAGNITUDE.

42. Let there be given the two sides DE, EC, in the triangle CDE (fig. 24), to find the hypothenuse, and one of the acute angles. It will soon be perceived, that the case now proposed cannot be solved by the table of sines and cosines. This table, indeed, may contain a triangle similar to the triangle proposed; from the data now given, however, no part of that triangle can be known, except the radius or hypothenuse; but neither of the sides DE, EC, can be compared with radius; by consequence from the parts now given the remainder cannot be determined.

43. For the solution of this case, it is then evident, that a series of triangles must be so constructed, that the radius shall be a side. In order to construct such a series, from the extremity of the radius AC (fig. 18) we raise the indefinite tangent AH, and draw from the centre C through each point in the quadrant AB the

secants CG, CG′, CG″, &c. It is evident, that the triangles CAG, CAG′, CAG″, &c. right angled at A, must have all the combinations of angles which can exist in a right angled triangle. The side AC moreover, common to all the triangles, will be the radius of the quadrant.

44. In the application of terms to express the different sides in the triangles of such a series, the common side AC, for the same reason as in the former case, may be called *radius*. The portions AG, AG′, AG″, &c. of the indefinite tangent AH, or in other words the perpendiculars in the triangles CAG, CAG′, CAG″, &c. are tangents of the arcs AE, AE′, AE″, &c; since we define the tangent of an arc, *a perpendicular drawn to the radius at one extremity of the arc, and terminated by the radius produced which passes through the other extremity.* The term *tangent* may, therefore, be applied to designate the perpendiculars of the triangles. The hypothenuses of the same triangles are secants of the arcs AE, AE′, AE″, &c; since we define the secant of an arc, *the radius drawn through one extremity of this arc, and produced until it meets the tangent drawn through the other extremity.* The hypothenuses of the triangles may in consequence be called *secants.*

45. We proceed to develop rules for the calculation of the sides in this second series of triangles.

The triangles CDE, CAG (fig. 18), having the angle C common, are similar; the sides of the latter may, therefore, be deduced from those of the former.

To find AG, we have CD : DE : : CA : AG;

that is, $\cos AE : \sin AE :: R : \text{tang } AE$;

hence $$\text{tang } AE = \frac{R \times \sin AE}{\cos AE}.$$

In order, therefore, to find the tangent of an arc, when the sine is given; *multiply the radius by the sine of the arc, and divide the product by its cosine.*

To find CG, we have $\text{CD} : \text{CE} :: \text{CA} : \text{CG}$;

that is, $\cos \text{AE} : \text{R} :: \text{R} : \sec \text{AE}$;

hence
$$\sec \text{AE} = \frac{\text{R}^2}{\cos \text{AE}}.$$

To find, therefore, the secant of an arc, the cosine being given; *divide the square of the radius by the cosine of the arc.*

46. From what has been said, it is evident, that having calculated a series of triangles in the quadrant of a circle, in which the radius is the hypothenuse of the triangles, we may, by the rules now obtained, calculate a second series of triangles similar to the former, in which the radius shall be a side. The parts in this second series may, moreover, be arranged in a table, as before; or which is the same thing, the table, already formed from the first series of triangles, may be extended by writing down against the several angles in this table the tangents and secants of these angles.

Below is a table of tangents and secants for every degree from 26° to 45° inclusive.

Deg.	Tangent	Secant	Deg.	Tangent	Secant
26	48773	1.11260	36	72654	1.23607
27	50953	1.12233	37	75355	1.25214
28	53171	1.13257	38	78129	1.26902
29	55431	1.14335	39	80978	1.28676
30	57735	1.15470	40	83910	1.30541
31	60086	1.16663	41	86929	1.32501
32	62487	1.17918	42	90040	1.34563
33	64941	1.19236	43	93252	1.36733
34	67451	1.20622	44	96569	1.39016
35	70021	1.22077	45	1.00000	1.41421

SECTION VIII.

SOLUTION OF TRIANGLES BY THE TABLE OF TANGENTS AND SECANTS.

47. To show the use, which may be made of the table of tangents and secants, from the point F (fig. 23) draw the perpendicular FG; then will FG be the tangent, and BG the secant of the angle B; BF will be the radius, and GBF the tabular triangle similar to ABC.

To find AC, we have BF : FG : : BC : AC;

that is, R : tang B : : BC : AC;

a proportion which we may thus enunciate; *radius is to the tangent of one of the acute angles in a right angled triangle, as the side of the right angle adjacent to this acute angle is to the side opposite.*

48. Ex. 1. In the triangle ABC (fig. 25), given the base BC 500 yards and the angle at B 32°, to find the perpendicular AC.

For this we have

R : tan B : : BC : AC,

or 100000 : 62487 : : 500 : 312.435 yds. Ans.

Ex. 2. Given the base BC 653 feet, and the perpendicular AC 547.932 feet, to find the angle at B.

BC : AC : : R : tan B,

653 : 547.932 : : 100000 : 83910.

From inspection of the table we find that 83910 in the column of tangents corresponds to 40°, which is the angle sought.

By means of the tangents, it is evident, we may determine one of these three things, when two of them are known; viz. *the two sides of a right angle and an acute angle.*

49. It remains to show the use, which may be made of the secants. In the triangle ABC (fig. 23) we have

BF : BG : : BC : BA; that is, R : sec B : : BC : BA,

a proportion, which we thus enunciate; *radius is to the secant of one of the acute angles in a right angled triangle, as the side of the right angle adjacent to this acute angle is to the hypothenuse.*

Ex. 1. In the triangle ABC (fig. 25) given the base 379 rods, and the angle B 35°, to find the hypothenuse.

R : sec. B : : BC : BA;

or 1.00000 : 1.22077 : : 379 : 462.67 ft. Ans.

Ex. 2. Given BA 245 rods, and the angle B 39°, to find BC.

sec B : R : : BA : BC;

or 1.28676 : 1.00000 : : 245 : 190.4 ft. Ans.

Ex. 3. Given BA 154.509, and BC 125, to find the angle at B. Ans. 36°.

By means of the secants the same things are determined, as by the sines and cosines; the secants are in consequence but seldom employed in the solution of triangles.

50. The preceding examples, it will be perceived, do not comprehend the case, in which any two of the sides are given to find the third. This case may be solved by means of the known property of a right angled triangle, viz. the square of the hypothenuse is equal to the sum of the squares of the two sides. It may, moreover, be solved with facility by means of the two propositions, art. 39, and 47.

In the use of these we find first an angle, and then the remaining side.

Ex. 1. In the triangle CDE (fig. 24), given the side DE 75, the side CE 50.59, to find the hypothenuse CD. Ans. 90.47.

Ex. 2. In the triangle CDE (fig. 24), given the hypothenuse CD 264, the side CE 135.97, to find the side DE. Ans. 226.28.

51. From what has been done, it is manifest that any right angled triangle whatever may be determined by means of a table constructed in the manner which has been described; provided that the triangle proposed be similar to some one of the triangles of the table, and that two of its parts be given, exclusive of the right angle. We pass to a few miscellaneous examples.

Ex. 1. Given the hypothenuse 435, and one of the acute angles 44°, to find the side opposite.

Ans. 302.177.

Ex. 2. Given the hypothenuse 64, and the base 51.778, to find the angle adjacent.

Ans. 36°.

Ex. 3. Given the base 200, and the adjacent angle 26°; required the perpendicular.

Ans. 97.546.

Ex. 4. Given the base 150, and the perpendicular 90.129; required the hypothenuse.

Ans. 174.99.

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SECTION IX.

OF COTANGENTS AND COSECANTS.

52. For the sake of convenience the table now constructed is usually still further extended, by writing down against the several angles in the table the tangents and secants of their complements.

53. If from the extremity of the radius BC (fig. 26), we draw the tangent BF to meet the secant CG, then, since the arc EB is the complement of the arc AE, BF will be the tangent, and CF the secant of the complement of the arc AE; BF may therefore, for the sake of brevity, be called the *cotangent*, and CF the *cosecant* of AE.

54. The cotangent and cosecant of the arc AE (fig. 26) evidently belong to a triangle distinct from that to which the tangent

and secant are found; the former of these, however, may be deduced from the latter. Indeed, on account of the similar triangles CAG, CBF, we have

$$GA : AC :: CB : BF; \text{ that is, } \text{tang } AE : R :: R : \text{cot } AE;$$

hence
$$\text{cot } AE = \frac{R^2}{\text{tang } AE}.$$

In order, then, to find the cotangent of an arc, when its tangent is given; *divide the square of the radius by the tangent of the arc.*

Again, to find CF, we have

$$GA : GC :: BC : CF; \text{ that is, } \text{tang } AE : \text{sec } AE :: R : \text{cosec } AE;$$

hence
$$\text{cosec } AE = \frac{\text{sec } AE \times R}{\text{tang } AE}.$$

To find, therefore, the cosecant of an arc, when its tangent and secant are given; *multiply the secant of the arc by radius, and divide the product by its tangent.*

55. To show the use, which may be made of the cotangents and cosecants, let there be given in the triangle ABC (fig. 27) the side AC 6 yds, the angle at B 50°, to find the remaining side and the hypothenuse. From the point A with an extent equal to one yard describe the arc MK; draw HM perpendicular to AC; AHM will be the tabular triangle similar to the triangle proposed.

1. To find CB we have

$$AM : MH :: AC : CB,$$
$$R : \text{cot } B :: AC : CB,$$
$$1.00000 : 83910 :: 6 : 5.0346.$$

2. To find AB,
$$AM : AH :: AC : AB,$$
$$R : \text{cosec } B :: AC : AB,$$
$$1.00000 : 1.30541 :: 6 : 7.8324.$$

3. Given AC 6, and AB 7.8324, to find the angle B.

56. In employing the cotangent and cosecant of B in the preceding example, it is evident, that we have merely used the tan-

gent and secant of A; the work is by consequence the same, as if we had found by subtraction the angle A, and then employed the tangent and secant of this angle.

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SECTION X.

MISCELLANEOUS REMARKS ON THE TRIGONOMETRICAL TABLES.

57. In the preceding articles we have shown the manner, in which a set of trigonometrical tables may be formed. Various methods of abridging the labor of calculation will naturally occur in practice. It is sufficient at present to remark, that from the nature of sines and cosines, tangents and cotangents &c. if the sines, cosines, tangents &c. have been calculated to 45° inclusive, the remainder of the series may be considered as known.

58. The several triangles of the tables for a given angle may be exhibited in connection with each other. In the triangle ABC (fig. 28) if we consider AC radius, BC will be the cotangent and AB the cosecant of the angle at B; if then with an extent equal to AC we draw from B the arc DF, and from the points D and F the lines DE, FG perpendicular to BC, then FG will be the tangent and BG the secant of the angle at B; DE, moreover, will be the sine and BE the cosine of the same angle; and the triangles ABC, GBF, DBE will be the tabular triangles for the angle at B.

59. The radius, it will be observed, assumes a different position in each of these triangles; in the triangle DBE, which contains the sine and cosine of the angle at B, it forms the *hypothenuse*; in GBF, which contains the tangent and secant of the angle at B, it becomes the *side adjacent* to this angle; and in ABC which contains the cotangent and cosecant of the same angle it becomes the *side opposite*. This circumstance must evidently be taken into consideration in calculating a proposed triangle by means of the tables. If the hypothenuse of the proposed triangle

be made radius, or if it be compared with the radius of the tables, the *sines* and *cosines* must be employed in the calculation of the triangle; if the side adjacent a given or required angle be made radius, the *tangents* and *secants* must be employed in the calculation; and if the side opposite the given or required angle be made radius, the *cotangents* and *cosecants* must be employed.

60. The numbers in the tables, the construction of which has now been explained, are usually denominated *natural* sines, tangents, &c. By means of these numbers, it has been seen that the sides and angles of any proposed triangle similar to some one of the triangles of the tables, may be accurately determined. But the calculations must be made in this case, by the tedious processes of multiplication and division. To avoid this inconvenience, another set of tables may be constructed by writing down against each of the angles the logarithm of its natural sine, tangent, &c. In the use of the numbers of such a set of tables, addition and subtraction will take the place of multiplication and division.

61. The logarithmic sines, tangents, &c. are commonly called *artificial* sines, tangents, &c. to distinguish them from the natural sines, tangents, &c. In comparing these two sets of tables, one circumstance requires consideration. The radius, to which the natural sines, &c. are calculated, is unity. The secants and a part of the tangents are, therefore, greater than a unit, while the sines and another part of the tangents are less than a unit. When the logarithms of these are taken, some of the indices, it is evident, will be *positive*, and others *negative*. To remedy, therefore, the inconvenience, which would result from the occurrence of both positive and negative indices in the same set of tables, ten is added to each of the indices, by which they are all rendered positive. Thus the natural sine of one minute is 0.00029, or more accurately 0.000290888; the logarithm of this is $\bar{4}.46373$; but the index by the addition of 10 becomes (10 — 4) or 6; by consequence the logarithmic sine of one minute is 6.46373. In like manner the logarithmic sine of two minutes is 6.76476, and so of the rest.

62. The learner may now solve, by aid of the logarithmic sines &c. the following cases.

Ex. 1. Given the hypothenuse of a right angled triangle 275, and the angle at the base 57° 23′; to find the remaining parts.

1. To find the perpendicular we have

Radius	10.000000
Is to the hypothenuse, 275,	2.439333
As sine 57° 23′,	9.925465
To the perpendicular, 231.63.	2.364798

2. In like manner we find the base 148.23.

Ex. 2. Given the base of a right angled triangle, 200 feet, and the adjacent angle 47° 30′, to find the perpendicular.

Ans. 218.26 feet.

Ex. 3. Given the hypothenuse 480, and the perpendicular 384, to find the angle opposite the perpendicular.

Ans. 53° 8′.

Ex. 4. Given the base 195, and the perpendicular 216, to find the hypothenuse. Ans. 291.

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SECTION XI.

SOLUTION OF OBLIQUE ANGLED TRIANGLES.

63. From what has been said, it is evident, that, in order to calculate a triangle by means of the tables, we must be able to construct from the numbers in the tables a triangle similar to the one proposed for calculation.

Let there be given the oblique angled triangle ABC (fig. 29, A). It is proposed to construct, from the numbers in the tables, a triangle similar to this. Having circumscribed a circle about the given triangle, from the centre D of this circle draw the radii DA, DB, DC; with the radius D*b*, equal to that of the tables, describe the circle *abc*; draw the chords *ab*, *bc*, *ac* joining the points of section *a*, *b* and *c*; since *Da* is equal to *Db*, the lines DA, DB are cut proportionally at the points *a* and *b*; *ab* is therefore parallel to AB; for a similar reason, *ac* is parallel to AC, and *bc* to BC; the triangle *abc* is, therefore, similar to the triangle ABC (Geom.). Upon the side *ab* of the triangle *abc* let fall from D the perpendicular D*e*. Since *aD* is equal to the radius of the tables, *ae* will be the sine of the angle *aDe*, and *be* the sine of the angle *bDe*. Produce *De* to meet the circumference of the circle *abc* in *h*. Since the arc *ah* is equal to the arc *hb*, the angle *aDe* is equal to the angle *bDe*; the side *ab* of the triangle *abc* is, therefore, equal to twice the sine of the angle *aDe*; or, since the angle *aDe* is equal to the angle *acb* (Geom.), the side *ab* is equal to twice the sine of the angle *acb*. In like manner *bc* is equal to twice the sine of the angle at *a*, and *ac* to twice the sine of the angle at *b*. But the angles *a*, *b*, and *c* in the triangle *abc* are equal respectively to the angles A, B, and C in the triangle ABC; the sides *ab*, *bc*, *ac* in the triangle *abc* are, therefore, equal respectively to twice the sines of the angles C, A, and B opposite the homologous sides AB, BC, AC in the triangle ABC.

With respect to the side *ab* in the case in which the angle C is greater than a right angle (fig. 29, B); since from the definition of a sine, an angle will have the same sine with its supplement to two right angles; *ae*, the sine of the angle *aDe*, will also be the sine of the angle *aDk*; and the side *ab* will be equal to twice the sine of the angle *aDk*. But the angle *aDk*, or, which is the same thing, the angle ADE is equal to the angle ACB; the side *ab* is, therefore, equal, as before, to twice the sine of the angle at C. From the numbers in the tables we may, then, always construct a triangle similar to any oblique angled triangle, which may be proposed for calculation, by taking for each of the sides in the required triangle twice the sine of the angle opposite the homologous side in the triangle proposed. The tables, therefore, which at first appear limited to the solution of right angled triangles, may evidently be applied to the solution of triangles of whatever kind.

64. We proceed to the investigation of rules for the calculation of plane triangles of whatever kind by means of the tables.

The similar triangles ABC, *abc* (fig. 29) give

$$AB : BC :: ab : bc \text{ \&c.};$$

that is, $AB : BC :: 2 \sin C : 2 \sin A$; or, $\sin C : \sin A$;

a proportion, which, it is easy to see, is of general application, and may be enunciated, as follows; *in any triangle whatever, the sines of the angles are to each other as the sides opposite to these angles.*

65. Case 1. To apply this principle, let there be given in the triangle ABC (fig. 30) the side BC 70 yds, the angle A 86°, and the angle C 45°, to find the remaining parts.

1. To find AB,	sin A 86°	9.998941
	BC 70 yds	1.845098
	sin C 45°	9.849485
		11.694583
	AB 49.62 yds.	1.695642

2. In like manner, AC is found equal to 52.96 yds.

Case 2. Given two of the sides, viz. AB 49.62 yds, BC 70 yds, and the angle A opposite BC 86°, to find the remaining parts.

1. To find the angle at C.

BC 70 yds	1.845098
sin A 86°	9.998941
AB 49.62 yds.	1.695642
	11.694583
sin C 45°	9.849485

2. Having obtained the angle C, the side AC is found as in the preceding case.

3. Given in the triangle ABC (fig. 13) the side AB 72 feet, the side AC 41.5 feet, and the angle B 30°, to find the angle C.

AC 41.5	1.618048
sin B 30°	9.698970
AB 72	1.857333
	11.556303
sin C 60° 10′	9.938255

In this last example the solution (art. 16) is ambiguous. Since, however, the angles ACB, AC′B are supplements, the one of the other, and the sine is the same for an angle and its supplement, if the angle 60° 10′ first obtained does not answer the conditions of the question, we then take its supplement 119° 50′ for the angle C which is sought.

66. From the preceding examples, it appears, that we may always solve a triangle by the rule just obtained, 1. *when a side and two of the angles are given; 2. when two of the sides and an angle opposite to one of them are given.*

67. In the triangle ABC (fig. 30) let there now be given the sides AC and BC, and the angle C contained between them, to find the remaining parts. This case, it will be perceived, does not admit of solution by the preceding rule. A new one must, therefore, be sought. Having described a circle with a radius equal to that of the tables (fig. 31), draw the diameter AM; let the arc AB, set off from A, be supposed equal to the number of degrees, contained in the angle at A in the proposed triangle; and also the arc AC to the number of degrees in the angle at B. From B draw the chord BD perpendicular to AM; from C draw CP perpendicular and CF parallel to AM; join BF, FD; from F, with the radius of the circle ABD equal to that of the tables, draw the arc IGK meeting CF in G; and at the point G draw HL perpendicular to CF. The line GL is the tangent of the angle GFL; or which is the same thing, of CFD; and GH is the tangent of GFH,

or CFB; and the angles, CFD, CFB having their summits in the circumference are measured by half the arcs CD, CB on which they stand. But CD is the sum of the two arcs AB, AC, and CB is the difference of these two arcs; whence GL is the tangent of the half sum, and GH the tangent of the half difference of the arcs AB, AC. The arcs AB, AC, by hypothesis, are respectively the measures of the angles A and B in the proposed triangle ABC (fig. 30); GL is therefore the tangent of the half sum and GH the tangent of the half difference of the angles A and B. In like manner it may be shown, that DE is the sum, and BE the difference of the sines of the same angles A and B; wherefore on account of the parallel lines BD, HL, we have

$$DE : BE :: LG : GH;$$

or putting for these lines what they severally denote

$$\sin A + \sin B : \sin A - \sin B :: \tan\frac{A+B}{2} : \tan\frac{A-B}{2};$$

but by art. 64 we have $\sin A : BC :: \sin B ; AC$; hence

$$\sin A + \sin B : \sin A - \sin B :: BC + AC : BC - AC;$$

whence, $BC + AC : BC - AC :: \tan\frac{A+B}{2} : \tan\frac{A-B}{2}$;

a proportion, which we may thus enunciate; *the sum of two sides of a triangle is to their difference, as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Ex. 1. Let AC (fig. 30) be 52. 96 yds, BC 70 yds, and the angle C 45°; it is required to find the remaining parts. Subtracting the angle C 45° from 180°, and dividing the remainder by 2, we have

$\frac{A+B}{2} = 67° 30'$; and to find $\frac{A-B}{2}$ we have

$BC + AC$	122.96	2.089763
$BC - AC$	17.04	1.231470
$\tan \frac{A+B}{2}$	67° 30′	10.382776
		11.614246
$\tan \frac{A-B}{2}$	18° 29′ 57″	9.524483

Having obtained the half difference between A and B, the greater angle A is found by adding the half difference to the half sum, and the less angle B by subtracting the half difference from the half sum. The angles being thus obtained, the remaining side may be calculated as before.

Ex. 2. Given two sides of a plane triangle 450 and 540, and the included angle 80°, to find the remaining parts.

Ans. Angles 56° 11′, 43° 49′, and the side 640.08.

Ex. 3. Given two sides of a plane triangle 76 and 109, and the included angle 101° 30′, to find the remaining parts.

Ans. Angles 30° 57′ 30″, 47° 32′ 30″; and side 144.8.

69. The preceding rules do not include the case, in which the three sides are given to find the angles. To obtain an expression for this case, in the triangle ABC (fig. 32) let fall from the point B, BD perpendicular to AC; and from the same point, with a radius equal to the side BC, describe the circumference CEHF; extend the side AB, until it meets the circumference in E; then, since AE and AC are drawn from the same point A without the circle,

$$AC : AE :: AG : AF,$$

(Geom.); but AE is equal to AB+BC, and AG is equal to AB—BC; AF, moreover, is equal to AD—DC; whence

$$AC : AB + BC :: AB - BC : AD - DC.$$

If, then, upon the longest side of a triangle a perpendicular be let fall from the opposite angle, we have the following proportion; *as the longest side of a triangle is to the sum of the two other sides,* *so is the difference of these last to the difference of the segments made by the perpendicular.*

70. Ex. 1. Given the three sides of a triangle ABC (fig. 33), viz. AB 66 feet, AC 75, and BC 34 feet, to find the angles. By the preceding rule,

AC 75	1.875061
AB + BC 100	2.000000
AB — BC 32	1.505150
	3.505150
AD — DC diff. seg. 42.67	1.630089

Adding half the difference of the segments to half the sum, we obtain the greater segment AD 58.83 feet; subtracting the half difference from the half sum, we obtain the less segment DC 16.17 feet.

Then to find the angle A, AB : R : : AD : cos A 26° 57′; and to find the angle C, BC : R : : DC : cos C 61° 36′.

Ex. 2. The sides of a plane triangle are 40, 34 and 25 feet respectively; required the angles.

Ans. 38° 25′ 20″, 57° 41′ 25″, 83° 53′ 15″.

Ex. 3. The sides of a plane triangle are 390, 350 and 270 feet respectively; required the angles.

Ans. 42° 21′ 57″, 60° 52′ 42″, and 76° 45′ 21″.

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SECTION XII.

MENSURATION OF HEIGHTS AND DISTANCES.

71. One of the most simple applications of Trigonometry is to the determination of the heights and distances of objects, when, by reason of intervening obstacles, a direct measurement cannot be made, or when it cannot be made with convenience.

Thus, let it be required to find the distance between the points A and B (fig. 30), which, by reason of some obstacle, we cannot measure directly.

The distance AB, it is evident, can be found, if we make it a side in a triangle, and determine a sufficient number of the remaining parts of the triangle for its calculation. Let us then, beginning at B, measure in some convenient direction a straight line BC of any convenient length; and at the extremities B and C of this line, measure the angles ABC, BCA. There will then be given three parts in the triangle ABC, by which the required part or distance AB may be found.

Ex. Suppose BC, the distance measured, 400 yards, the angle ABC 73° 15′, the angle BCA 68° 2′; what is the distance between the points A and B?

Subtracting the sum of the angles ABC, BCA from two right angles we have $\text{BAC} = 38° \ 43'$; then

$$\sin \text{BAC} : \text{BC} :: \sin \text{BCA} : \text{AB} = 593.09 \text{ yds. Ans.}$$

The lines and angles which it is required to measure, in order to the determination of heights and distances, are situated in planes which are designated by particular names, according to their position.

The earth being considered as a sphere, a plane perpendicular to a radius at any point of its surface, or any plane parallel to this, is called a *horizontal plane.*

A plane perpendicular to a horizontal plane is called a *vertical plane;* and a plane oblique to it is called an *oblique plane.*

Lines situated in a horizontal plane are called *horizontal lines.* Those perpendicular to a horizontal plane are called *vertical lines;* and lines inclined to a horizontal plane are called *oblique lines.*

Angles the sides of which are situated in a horizontal plane are called *horizontal angles.* If the sides are situated in a vertical plane they are called *vertical angles;* if in an oblique plane they are called *oblique angles.*

A vertical angle contained between a horizontal and an ascending line, is called an angle of *elevation*. A vertical angle contained between a horizontal and a descending line, is called an angle of *depression*.

The principal line measured, for the purpose of determining a height or distance is called a *base line*, or more simply the *base*. For the measurement of the lines required, we may employ any of the usual linear measures; as the foot, yard, &c.

The instrument usually employed for the measurement of the necessary angles, is called a *Theodolite*. It is adapted to the measurement both of horizontal and vertical angles. Its principal parts are, therefore, a graduated circle, with means of adjustment by which it can be placed in a horizontal plane, for the measurement of horizontal angles; and a graduated circle or part of a circle, the plane of which is at right angles with the former, for the measurement of vertical angles.

The following are among the problems which most frequently occur in practice.

PROBLEM I.

72. To determine the altitude of a steeple or other object situated on a horizontal plane.

From the bottom of the steeple (fig. 37) measure in a direct line upon the horizontal plane, any convenient base BA; at the extremity A of this line measure the angle BAC, comprised between the line BA and an imaginary line, drawn from the point A to C the top of the steeple; then in the triangle ABC the side AB and all the angles will be known; whence to find BC, we have

$$\sin ACB : AB :: \sin BAC : BC;$$

or otherwise $$R : \tan BAC : AB : BC.$$

Ex. 1. Let the base line AB be 200 feet, and the angle of elevation BAC be 47° 30′; required the height of the steeple.

Ans. 218.26 feet.

Ex. 2. Let AB be 384 feet, and the angle at A 36° 52′; required the height of the steeple. Ans. 288 feet.

PROBLEM II.

73. To determine the perpendicular height of a cloud, or other object, above a horizontal plane.

At two convenient stations A and B on the same side of the object (fig. 38), or on opposite sides (fig. 39), and in the same vertical plane, let two observers take at the same time the angles of elevation CAB, CBD, and let the distance AB be measured; then the exterior angle of a triangle being equal to the two interior and opposite, subtracting (fig. 38), the angle CAB from CBD, we have the angle ACB; whence

sin ACB : AB : : sin CAB : BC;

then, in the triangle CBD, to find CD the height required, we have

R : BC : : sin CBD : CD;

or (fig. 39) subtracting the sum of the angles at A and B from two right angles, we have, the angle ACB; then to find one of the remaining sides in the triangle ACB, AC for example, we have

sin ACB : AB : : sin ABC : AC;

and in the triangle ADC, to find CD we have

R : AC : : sin CAD : CD.

Ex. 1. Let the angles of elevation (fig. 38) be 31° and 46° respectively, and the base AB, 100 yards; what is the height of the cloud? Ans. 143.14 yds.

Ex. 2. Let the angles of elevation (fig. 39) be 53° and 79° 12′ respectively, and the distance AB 100 rods; what is the altitude of the cloud? Ans. 105.89 rods.

PROBLEM III.

74. To find the height of an inaccessble object standing on a horizontal plane.

At two stations A and B in the same vertical plane passing through the top of the object (fig. 40), take the angles of elevation CAB, CBD, and measure the distance AB; then in the triangle CAB we have

$$\sin ACB : AB :: \sin CAB : CB;$$

whence to find CD the altitude required, we have in the triangle CBD

$$R : CB :: \sin CBD : CD.$$

Ex. 1. Let the angles of elevation be 32° and 58°, and the base line 100 yards; required the height of the object.

Ans. 102.51 yards.

Ex. 2. Let the angles of elevation be 40° and 60° and the base line 100 feet; required the height of the object.

Ans. 162.75 feet.

PROBLEM IV.

75. To find the height of an accessible object standing on an inclined plane.

From the bottom of the object (fig. 41) measure in the same direct line any two distances AB, BD; at B take the angle CBA, and at D the angle CDA; then in the triangle CDB we have

$$\sin DCB : DB :: \sin CDB : BC.$$

BC being thus determined, we have in the triangle CBA

$$BC + BA : BC - BA :: \text{tang } \tfrac{1}{2} (A + C) : \text{tang } \tfrac{1}{2} (A - C);$$

having found by means of this proportion one of the remaining angles in the triangle CBA, BCA for example, to find the height required, we have

$$\sin BCA : BA :: \sin ABC : AC.$$

Ex. Let the distances AB, BD be 40 and 60 feet, the angle at B 41°, that at D 23° 45′; required the height of the object.

Ans. 57.624 feet.

PROBLEM V.

76. To find the distance between two objects inaccessible by reason of an intervening river, or other impassable barrier, the objects being both on the same level.

At two stations A and B (fig. 42) also on a level, we take the angles CAB, DAB, CBA, DBC and measure the distance AB; then in the triangle ACB we have

$$\sin ACB : AB :: \sin CAB : CB;$$

and in the triangle ABD

$$\sin ADB : AB :: \sin DAB : DB;$$

then in the triangle CBD we have

$$BC + BD : BC - BD :: \text{tang } \tfrac{1}{2}(D + C) : \text{tang } \tfrac{1}{2}(D - C);$$

having found by means of this proportion the angle BCD, for example, we have finally for the distance sought

$$\sin BCD : BD :: \sin CBD : CD.$$

Ex. Let the angles in the order stated above be 37°, 58° 20′, 53° 30′, 45° 15′, and the base line 300 yards. What is the distance between the two objects? Ans. 479.79 yards.

PROBLEM VI.

77. The height of a tower being given to find the horizontal distance between two objects, situated on the same level with the tower and in a direct line from the bottom of it.

Take the angles EAD, EAC (fig. 43) contained between the line EA parallel to the horizon and lines drawn from A, the top of the tower, to the objects D and C respectively; then in the triangle ADB all the angles and the side AB will be known, from which BD may be found; in like manner in the triangle ACB all the angles and the side AB will be known from which BC may be found; the value of these being determined, the latter subtracted from the former will give the distance sought.

Ex. Let the height of the tower be 120 feet, the angle of depression of the nearest object 57°, that of the most remote 25° 30′. What is the distance between the two objects?

Ans. 173.65 feet.

PROBLEM VII.

78. To find the altitude of a hill above a horizontal plane which passes through a given point on the side of another hill opposite.

At the given point B (fig. 45) take the angle of elevation CBD of the top of the hill, the altitude of which is sought; from B measure a base line BA directly up the other hill, and at A take the angle of elevation CAE and the angle of depression EAB; then in the triangle CAB all the angles will be given and the side AB, to find CB; and CB being found, we shall have in the triangle CBD all the angles and the side CB, to find CD the altitude sought.

Ex. Let the angle CBD be 5° 52′, the base line AB 642 yards, and the angles CAE, EAB 3° 59′, and 0° 39′ respectively. What is the altitude of the hill?

Ans. 161.3 yards.

PROBLEM VIII.

79. To find the distance between two inaccessible objects, both of which can be seen at the same time from one point only.

Let D (fig. 46) be the point from which both objects can be seen at the same time; in any convenient directions take a station C where A can be seen, and a station E where B can be seen, and measure the distances CD, DE; at D take the angles ADC, ADB, BDE, and at C and E the angles ACD, BED; then in the triangles ACD, BDE we shall be able to determine the sides AD, DB; and these being found, we shall have in the triangle ADB data sufficient to determine AB the distance sought.

Ex. Let CD be 200 yards, the angle ADC 89°, ACD 50° 30′; let DE also be 200 yards, the angle BDE 54° 30′, BED 88° 30′, and ADB 72° 30′; required the distance between the two objects.

Ans. 345.5 yards.

PROBLEM IX.

80. To find the altitude of an inaccessible object situated on an elevation, the observer being on a horizontal plane.

Having assumed a station B (fig. 47) upon the horizontal plane; measure in a direct line from the object a base line BA; at B take the angles of elevation DBE, CBE, and at A the angle of elevation CAE. In the triangle CAB we determine the side CB; then in the triangle CBE we find the remaining sides CE and BE; this being done, in the triangle DBE we find the side DE; subtracting next DE from CE already found we have the altitude CD sought. CD may also, it is evident, be found by means of the triangle CBD without calculating CE and DE.

Ex. Let the angle DBE be 40°, CBE, 51°, the base line BA 100 yards, and the angle CAE, 33° 45′, to find the altitude of the object.

Ans. 46.67 yards.

PROBLEM X.

81. To determine the distance of an inaccessible object, when an instrument for measuring angles cannot be procured.

Let E be the object (fig. 50) and AE the distance sought. From the station A in the direction EA, measure any convenient distance AC; assume another station B, and in the direction EB measure any convenient distance BD; measure also the distances AB, AD and BC. In the triangles ACB, ABD we shall have the sides given to determine the angles; and these being found, those of the triangle AEB will be known; then in the triangle AEB we have the angles and a side to find AE the distance sought.

Ex. Let AB be 500, AC 100, CB 560, BD 100, and AD 550 yards; required the distance AE.

Ans. 536.25 yards.

PROBLEM XI.

82. To find the distance of a heavenly body from the earth.

The distances thus far determined relate to objects upon or near the surface of the earth. Let it now be proposed to determine the distance of any one of the heavenly bodies from the centre of the earth.

The semi-diameter of the earth being known, we take this for the *base line.*

Let A be the place of the observer upon the earth's surface, C the centre of the earth, P the place of the heavenly body, considered as in a plane passing through the centre C of the earth perpendicular to the radius AC. Join AP, and CP; and let AQ be a horizontal line drawn through A, and parallel to CP. The angle QAP, contained between this line and AP, is called the horizontal *parallax* of the heavenly body P; and may be found by observation. This being done, we shall have two of the angles and a side in the right angled triangle ACP, to find the distance CP sought.

Ex. The horizontal parallax of the moon is found by observation to be 0° 57′, and the diameter of the earth is 7940 miles; what is the distance of the moon from the centre of the earth?

Ans. 239414 miles.

The distance of a heavenly body from the earth being found, its magnitude may also be determined, if it have an apparent diameter sufficiently large to be measured by an instrument for the measurement of angles.

Let A be the place of the observer, C the centre of the heavenly body, and AC its distance from the earth. Suppose two tangent lines AM, AM′ to be drawn in the plane of AC and touching the body in the points M, M′; and join MC, M′C. If now the magnitude of the angle MAM′ can be determined by observation, the magnitude of the body may be found. Indeed, we have for this purpose the right angled triangle AMC, right angled at M, and in which the side AC, and the angle MAC are known; and from which the semi-diameter, and by consequence the magnitude of the body may be found.

Ex. 1. The sun subtends at the earth an angle of 32′ 2″, and his distance from the earth is 95000000 miles. What is his diameter? Ans. 885000 miles.

Ex. 2. If the moon subtends an angle of 31′ 14″, when her distance from the earth is 240000 miles; what is her diameter?

Ans. 2180 miles.

QUESTIONS AND PROBLEMS FOR PRACTICE.

83. From what has been done the learner will see the manner in which the principles are applied to the object proposed. We subjoin as an exercise a few additional questions and problems.

1. A line 27 yards long will exactly reach from the top of a fort to the opposite bank of a river, known to be 23 yards broad; what is the height of the wall? Ans. 42.42 feet.

2. The height of an object on a horizontal plane is 200 feet, and its angle of elevation at the place of the observer is 42° 30′; what is his distance from the object? Ans. 218.26 feet.

3. A ladder 193.55 feet long is placed against a wall in an oblique position, and reaches to a point 75.83 feet from the ground; what is the angle at which it is inclined to the ground?

Ans. 23° 3′ 55″.

4. From the edge of a ditch 18 feet wide, surrounding a fort, I took the angle of elevation of the top of the wall and found it 62° 40′; required the height of the wall, and the length of a ladder necessary to reach from my station to the top of it.

Ans. Height 34.82, length 39.2 feet.

5. A ladder 125 feet long is placed against a wall so that the angle at the bottom is double the angle at the top; how high up the wall does it reach and how far distant from the wall is its foot?

Ans. 108.25, and 62.5 feet respectively.

6. A person attempts to swim across a river, whose breadth is 329 yards. How far from the place immediately opposite to his departure does he arrive, if the stream forces him to swim twice as far as he would have done, had there been no current?

Ans. 569.9 yards.

7. From the top of a tower, 143 feet high, by the sea side, I observed that the angle of depression of a ship's bottom, then at anchor, was 55°; what was its distance from the bottom of the wall? Ans. 100.13 feet.

8. Two ships of war wishing to ascertain their distance from a fort, sail from each other a distance of half a mile, when they find that the angles formed between a line from one to the other, and from each to the fort, are 85° 15′, and 83° 45′. What are their respective distances from the fort?

Ans. 4584.52, 4596.10 yards.

9. Wanting to know the breadth of a river, I measured 100 yards in a right line close by one side of it, and at each end of this line I found the angles subtended by the other end and a tree close by the other side of the river to be 53° and 79° 12′. What is the perpendicular breadth? Ans. 105.89 yards.

10. Being on one side of a river and wanting to know the distance to a house, which stood on the other side, I measured 200 yards in a right line by the side of the river, and found that the two angles at each end of this line formed by the other end and the house were 73° 15′, and 68° 2′; what was the distance between each station and the house? Ans. 296.54 and 306.19 yards.

11. A gentleman wishing to ascertain the distance between two trees A and B, which could not be directly measured on account of a pool, that occupied the intermediate space, assumed a station C, from which both could be seen, and found by measurement the distance AC 7.35 chains, the distance BC 8.4 chains, and the angle ACB 55° 40′. What is the distance of the trees A and B?

Ans. 7.412 chains.

12. From the top of a tower, whose height is 108 feet, the angles of depression of the top and bottom of a vertical column standing in the horizontal plane are found to be 30° and 60° respectively; required the height of the column.

Ans. 72 feet.

13. From a window B near the bottom of a house supposed to be on a level with a monument CD, the angle of elevation CBD of the top of the monument being 40°, and from another window A 18 feet higher the angle of elevation CAE being 37° 30′, it is required to find the height of the monument CD and its distance BD.

Ans. CD 210.44 feet, and BD 250.79 feet.

14. Wanting to know the height and distance of an object on the other side of a river, and on a level with the place where I stood, close by the side of the river; not having room to go backward on the same plane, on account of the immediate rise of the bank, I placed a mark where I stood and measured in a direct line from the object up the hill a distance of 132 yards; I then found the angle of depression of the mark by the river's side 42°, that of the bottom of the object 27°, and of its top 19°; required the height of the object, and the distance of the mark from its bottom.

Ans. Height 28.63, distance 75.25 yards.

15. It is required to find the height of a castle AB, situated on an eminence by the sea shore, above the level of the sea and its horizontal distance from a ship S at anchor, the angles of depression, EAS, FBS being given equal respectively to 4° 52′, and 4° 2′, and the height of the castle being 54 feet.

Ans. Distance 3690 feet, and height 314 feet.

16. At a station A at the bottom of a hill, I took the angle of elevation of the top of an object D on the summit of the hill, 38°, and measuring directly up the hill a distance of 676.47 feet to another station B, I took at this station the angle of elevation of the top of the object D, 46°, and the angle of depression of the first station A, equal to the angle of elevation of the second station

from the first, 27° 30′ 25″; what is the height of the object D above the level of the first station A. Ans. 949.19 feet.

17. Suppose the object CD (fig. 49) to stand upon a horizontal plane ABD and that AB is equal to 250 yards, and that the angles at its extremities are known, viz, CAB 56° 46′, CBA 62° 54′, DAC 6° 40′, DBC 7° 6′. What is the height CD and the two distances AD and BD?

Ans. AD 254.989, BD 238.814, and CD 29.745 yards.

18. The shadow of the top of a mountain, when the altitude of the sun on the meridian is 32°, strikes a certain point on a level plane below; but when the meridian altitude of the sun is 67° the shadow strikes half a mile farther south, on the same plain. What is the height of the mountain above the plain?

Ans. 2245 feet.

19. To ascertain the distance between two places, the angle which they subtended at a point equally distant from both was observed, and again at another point equally distant from both; and the distance between the points of observation was measured. To determine from these data the distance of the places, 1°, when the points of observation are both on the same side of the places; 2°, when they are on opposite sides.

20. The altitude, or angle of elevation of a cloud, was observed to be $a°$, and that of the sun in the same direction $b°$; and the distance of the shadow of the cloud from the station of the observer measured c yards. Show how the height of the cloud may be found, the distance d of the sun being also given.

21. At a given distance from an obelisk, whose height is known, a colossal statue on the top of it subtends the same angle as the observer, when seen from the base of the obelisk. Supposing the obelisk and the observer on the same horizontal plane, and the height of the observer known, show how the height of the statue may be determined.

22. A flag-staff is placed upon a wall a feet long, in such a

situation that a line of b feet in length will reach from its top to one end of the wall, and a line of c feet from its top to the other. Determine the height of the flag-staff, and its distance from the ends of the wall.

23. Wishing to know the distance between two inaccessible objects A and B, and not finding any point from which both could be seen at the same time, two stations C and D were chosen at a given distance from each other, from the former of which A could be seen, and from the latter of which B could be seen. From C, I next measured a given distance CF, and from D a given distance DE, each in a direction different from CD. The following angles were then measured, viz, AFC, ACF, ACD; and BDC, BDE, BED. How may the distance between the two objects A and B be found from these data?

24. Two spectators at two different stations on a horizontal plane, the distance between which is known, observe at the same instant a balloon, which rises in the air at a uniform rate, and after a given interval of time again observe it; what are the observations necessary to determine the two elevations of the balloon and its rate of moving, and how shall the data be applied to this purpose, 1°. when it ascends in a perpendicular line and the stations are not in the same vertical plane with it; 2°, when it ascends in an oblique line and the stations are in the same vertical with it.

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SECTION XIII.

LINE OF NATURAL SINES, TANGENTS, &C.

84. Let the quadrant AB (fig. 35) be divided into equal portions, of a degree each, for example; and let lines be drawn, parallel to AC, from each point of division in the quadrant AB to the line CB. The several distances from C to the points, where these parallels meet the line CB, will be equal respectively to the lengths

of the sines for each degree in the quadrant AB. Thus the extent from C to 10 will be equal to the sine of 10°, from C to twenty to the sine of 20°, and so on. The line CB, divided in this manner, is called *a line of sines.*

In like manner if through each of the points of division in the quadrant AB secants be drawn to meet the indefinite tangent BD, the several distances from B to the points of division in BD will be equal respectively to the tangents for each degree in the quadrant AB, and the line BD will be a *line of tangents.*

If moreover the different distances from the centre C to each of the divisions on the line of tangents BD be transferred to the line CE, the line CE will be a *line of secants.*

Thus B 20 is the tangent, and C 20 the secant of 20°.

85. A single example will show the use, which may be made of these lines.

In the triangle ABC (fig. 30), given the side BC 70 yards, the angle A 86°, and the angle C 45°, to find AB.

By art. 65, sin A 86° : sin C 45° : : BC : AB. In order therefore to determine the value of AB by the line of sines, we measure by means of a convenient scale of equal parts, the extent from C to 86 on the line of sines (fig. 35), and also the extent from C to 45 ; we then have the relative values of the first and second terms in the above proportion.

Let the sine of A determined in this manner be 332, and the sine of C 235, then

332 : 235 : : 70 : 49.5 the side AB, &c.

The method of solving triangles, which has now been exhibited, is obviously the same, as that by the table of natural sines, &c., with the exception that the values of the sines in the present case are determined by geometrical construction.

SECTION XIV.

LINES OF LOGARITHMIC SINES, TANGENTS, &C.

86. The logarithms of numbers may be conveniently represented by lines. Thus, since the logarithm of 10 is 1 and the logarithm of 100 is 2, if a line of one foot be made to represent the logarithm of 10, a line of two feet will represent the logarithm of 100,

For the purpose of forming a logarithmic scale let the line *ab* (fig. 36), intended for the scale and taken of any length at pleasure represent the logarithm of 100; then 100 will stand at the end of the scale; and 1, the logarithm of which is 0, will stand at the beginning of the scale. For the intermediate numbers 1, 2, 3, 4, &c, it is easy to see, that we have merely to set off in order from the beginning of the scale the several lengths, which shall be equal respectively to the logarithms of these numbers, when *ab* is equal to the logarithm of 100. The logarithm of 100, taken from a common table of logarithms but extending to three places only of decimals, is 2.000, or, omitting the separatrix, 2000. Under the same circumstances the logarithm of 2 is 301, and the logarithm of 3 is 477, &c. Let it be supposed, that the line *ab* is divided into 2000 equal parts; then, as *ab* has been made to represent the logarithm of 100, 301 of these parts will represent the logarithm of 2. To determine therefore the place of the number 2 on the scale, we set 301 of the parts, into which *ab* is supposed to be divided, from 1 to 2; to determine the place of 3 we set off in like manner 477 of the same parts, and so of the rest. Proceeding in this manner we determine the primary divisions of the scale; and to obtain the intermediate divisions we set off in a similar manner the logarithms of the intermediate numbers. Thus, when the logarithm of 100 is 2000, the logarithm of 1.1 is 41, the logarithm of 1.2 is 79, &c. These numbers being set off in order from the beginning of the scale will divide the first primary division into 10 parts. In like manner the remaining primary divisions may be further divided.

87. The logarithmic scale, the construction of which has now been explained, contains the logarithms of numbers from one to a hundred only. The decimal part of the logarithm of any number, it will be recollected, is the same as that of the number, when multiplied or divided by 10 or 100, &c. We may then consider the point under the 1 at the beginning of the scale to represent the logarithm of 1, or 10, or 100, or $\frac{1}{10}$, or $\frac{1}{100}$, &c. Hence the same scale may easily be made to answer for all numbers whatever.

From what has been said the logarithm of any given number may with facility be taken from the scale.

Let it be required to take from the scale the logarithms of the following numbers, viz. 75, 43, 5, 365, 2450, 10844.

88. Multiplication, division, &c. are performed by the line of numbers on the same principles as by common logarithms.

Multiplication.

1. Let it be required to multiply 6 by 8. Since addition in logarithms takes the place of multiplication, the extent from 1 to 6 added to the extent from 1 to 8 will be equal to the extent from 1 to 48 the product.

To multiply then by the logarithmic line, we take off with the compasses that length of line which represents the logarithm of *one* of the factors, and apply this so as to extend forward from the end of that which represents the logarithm of the *other* factor. The sum of the two will reach to the end of the line, which represents the logarithm of the product.

2. It is required to multiply 9 by 7, 18 by 5, 44 by 63, 120 by 75.

Division.

It is required to divide 56 by 7, 120 by 12, 400 by 50, 2880 by 320.

Proportions.

1. To find the fourth term of the proportion, in which 2, 4 and 8 are the first three terms.

2. To find the fourth term of the proportion, in which 15, 75 and 40 are first three terms.

3. Let 4324, 540.5 and 2560 be the three first terms of a proportion; required the fourth term.

89. We proceed to construct a line of logarithmic sines, that shall correspond with the line of numbers. For this purpose let the line *cd*, intended for the scale, be taken (fig. 36) equal to *ab*. In order that this scale may correspond with the line of numbers, it will be obvious, that the difference between the extreme indices, on both should be the same. On the line of numbers the difference between the extreme indices is 2. The logarithmic sine of 0° 34′ 22″ 41‴, is 8.000, and that of 90 is 10.000, the difference of the indices being 2. If then the point *c* at the beginning of the scale be marked for the place of 0° 34′ 22″ 41‴, the point *d* at the end of the scale will be the place for 90°. In order to find the intervening divisions we suppose the line *cd*, in the same manner as has been supposed with respect to *ab*, to be divided into 2000 equal parts. Then, since the difference between the logarithmic sine of 0° 34′ 22″ 41‴ and the logarithmic sine of 1° is 241, the extent from *c* to 1 equal to 241 of the parts, into which *cd* is supposed to be divided, will give the point of division for the sine of 1°; in like manner the extent from 1 to 2, equal to 542 of the same parts, will give the point of division for 2°, and so of the rest. The scale may also be constructed by marking the point *d* for the sine of 90°, and setting the arithmetical complement of each degree &c. backward from *d* toward *c*.

90. The logarithmic tangent of 0° 34′ 22″ 35‴, is 8.000, and that of 45° is 10.000. To form therefore a line of logarithmic tangents corresponding to the line of numbers, let the point *e*, on

the line *ef* taken equal to *ab* be marked for the tangent of 0° 34′ 22″ 35‴, and the point *f* for the tangent of 45°; then, *ef* being supposed also to be divided into 2000 equal parts, the intervening points of division may be marked as before.

For the points of division above 45° the scale should extend much further to the right; but (art. 54) $\frac{\text{R}}{\text{tang}} = \frac{\text{cot}}{\text{R}}$; that is, in logarithms R — tang = cot — R. The logarithmic tangent, therefore, of an arc below 45° is as much less than that of 45°, as the logarithmic cotangent of this arc is greater than that of 45°. Instead, then, of extending the scale further to the right, the numbering after reaching 45° may be continued back from right to left, and the same point of division be made to answer for an arc and for its complement.

91. We shall now show the use, which may be made of the logarithmic lines in the solution of triangles.

Ex. 1. In the triangle ABC (fig. 30), given the side AB 40 yds, the side AC 90 yds, and the angle C 20°, to find by means of the logarithmic lines the angle B.

By art 65, AB : AC : : sin C : sin B; wherefore in logarithms AC — AB = sine B — sine C; hence to find the angle at B, we take in the compasses from the line of numbers the extent from 40 to 90; this extent will reach from 20° on the line of sines to 50° 20′ the answer.

2. Let the angle C equal 55°, the angle B 32°, and the side AB 100 feet to find AC.

3. Given AB 53, BC 13, and the angle B, 120° to find AC.

92. The scale fig. 36, the construction of which has now been explained, is commonly called *Gunter's scale*, from the name of its inventor. It furnishes a ready method of solving triangles in those cases, where great accuracy is not required. The mode of solution differs from that by the common tables of logarithms in this respect only, that the logarithmic values of the sides and angles are expressed by lines instead of numbers.

SECTION XV.

AREA OF TRIANGLES.

93. We have seen the manner in which the remaining parts of a plane triangle may be found, by means of certain parts which are given. We have seen also how the plane triangle may be employed in the determination of heights and distances. To complete the theory of the triangle it remains only to show how its area may be computed from the parts which are given.

In the triangle ABC (fig. 33) let us designate by a, b, and c, respectively, the sides BC, AC, AB opposite the angles A, B, and C. Let h denote the altitude BD; and let A represent the area required.

94. Prob. I. Given two sides and the included angle to find the area of the triangle.

Let AC, and BC be the two given sides, and ACB the included angle; then to find BD, we have

$$\text{R} : \sin \text{C} :: \text{BC} : \text{BD},$$

or, using the notation above, and making $\text{R} = 1$,

$$1 : \sin \text{C} :: a : h;$$

or
$$h = a \sin \text{C};$$

whence, multiplying the value of h by $\frac{1}{2}$ AC $= \frac{1}{2} b$, we have

$$\text{A} = \tfrac{1}{2} ab \sin \text{C}.$$

To find the area, therefore, when two sides and the included angle are given; *multiply half the product of the two sides by the sine of the included angle.*

Ex. 1. What is the area of a triangle in which two of the sides are 30 and 40, and the included angle 28° 57′?

$$\begin{array}{ll} \log 2 \text{ Comp} & = 9.698970 \\ \log 30 & = 1.477121 \\ \log 40 & = 1.602060 \\ \sin 28^\circ\ 57' & = \underline{9.684887} \\ & 22.463038 \end{array}$$

Deducting 20 from the characteristic we have

$$\text{Log } 2.463038 = 290.4276 \quad \text{Ans.}$$

In the above example we have used the arithmetical complements of the factors of the divisor, in order, Alg. art. 211, that the whole work may be performed by the addition of logarithms.

Ex. 2. What is the area of a triangle in which two of the sides are 45 and 32 feet, respectively, and the included angle 46° 30′ ?

Ans. 522.266 feet.

95. Prob. II. Given two angles and the included side, to find the area of the triangle.

Let AC (fig. 33) be the given side, A and C the given angles. Employing the notation above, we have

$$h = c \sin \mathrm{A};$$

but
$$c = \frac{b \sin \mathrm{C}}{\sin \mathrm{B}} = \frac{b \sin \mathrm{C}}{\sin (\mathrm{A} + \mathrm{C})},$$

the sine of an angle being the same with that of its supplement. Hence by substitution

$$h = \frac{b \sin \mathrm{A} \sin \mathrm{C}}{\sin (\mathrm{A} + \mathrm{C})};$$

whence, multiplying by $\frac{1}{2}b$, we obtain

$$\mathrm{A} = \frac{b^2 \sin \mathrm{A} \sin \mathrm{C}}{2 \sin (\mathrm{A} + \mathrm{C})}.$$

To find the area, therefore, when two angles and the included side are given; *Multiply together the square af the given side and the sines of the given angles; and divide the product by twice the sine of the sum of the given angles.*

Ex. 1. What is the area of a triangle in which one of the sides is 120 feet, and the adjacent angles 40° and 60°, respectively?

$$\begin{aligned} \log b^2 = 2 \log 120 &= 4.158362 \\ \sin 40^\circ &= 9.808067 \\ \sin 60^\circ &= 9.937531 \\ \log 2 \text{ Comp} &= 9.698970 \\ \sin 100^\circ \text{ Comp} &= \underline{0.006649} \\ & 33.609579 \end{aligned}$$

or subtracting 30 from the characteristic, on account of complements and tens added,

$$\log 3.609579 = 4068.5 \text{ square feet. Ans.}$$

Ex. 2. What is the area of a triangle in which one of the sides is 320 yards, and the adjacent angles 30° and 40° respectively?

96. Prob. III. Given the three sides, to find the area of the triangle.

In the triangle ABC (fig. 33) we have, using the same notation as above, $DC = a \cos C$; whence

$$h^2 = a^2 - a^2 \cos C^2. \qquad (1)$$

But by Geometry we have

$$AB^2 = AC^2 + BC^2 - 2\ AC \times DC;$$

or

$$c^2 = a^2 + b^2 - 2ab \cos C;$$

whence

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab};$$

and by substitution in (1)

$$h^2 = a^2 - \left(\frac{a^2 + b^2 - c^2}{2\,b}\right)^2. \qquad (2)$$

Extracting next the square root of this last, and multiplying by $\frac{1}{2}\,b$, we obtain an expression for the area. But the result thus obtained not being adapted to calculation by logarithms, we proceed as follows.

The two terms of the right hand member of (2) freed from the denominator, being the difference between two squares, we have

$$\begin{aligned} 4\,b^2h^2 &= (2\,ab + a^2 + b^2 - c^2)\,(2\,ab - a^2 - b^2 + c^2) \\ &= (\,(a+b)^2 - c^2)\,(c^2 - (a-b)^2)\,; \end{aligned}$$

but the two factors of this last also being each the difference between two squares, we have

$$4b^2h^2 = (a + b + c)\,(a + b - c\,)\,(c + a - b)\,(c + b - a);$$

hence, adding and subtracting c, b and a in the second, third and last factor respectively, and designating $a+b+c$ by $2s$, we have

$$4\,b^2h^2 = 2s\,(2s-2c)\;(2s-2b)\;(2s-2a)$$
$$= 16s(s-c)\;(s-b)\;(s-a).$$

Dividing next both sides by 16 and extracting the square root,

$$\frac{bh}{2} = (s(s-c)(s-b)\;(s-a)\;)^{\frac{1}{2}}.$$

Hence, to find the area of a triangle when the three sides are given; *Subtract successively each side from the half sum of the sides; multiply together the half sum and the three remainders; and take the square root of the product.*

Ex. 1. Given three sides 49, 50.25, and 25.69 feet, to find the area.

$$\begin{array}{lr} \text{log half sum, } 62.47 & = 1.795672 \\ \text{“ 1st remainder, } 13.47 & = 1.129368 \\ \text{“ 2d \quad “ \qquad } 12.22 & = 1.087071 \\ \text{“ 3d \quad “ \qquad } 36.78 & = \underline{1.565612} \\ & 2\,\underline{)\;5.577723} \\ \text{log } 615.75 \text{ square feet} & = 2.788861 \quad \text{Ans.} \end{array}$$

Ex. 2. What is the area of a triangle whose sides are 125, 173, and 216 feet?

Ans. 10809 square feet.

BOOK II. SURVEYING.

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SECTION I.

DEFINITIONS. DESCRIPTION OF INSTRUMENTS.

97. We have seen the manner in which the distances between objects on the surface of the earth, and by consequence, their relative position, may be determined, by means of certain lines and angles which are measured.

The determination of heights and distances belongs to the more general subject of *Surveying*, which in its widest signification, embraces the following particulars.

1°. To perform the operations necessary to determine the outlines and principal features of some limited portion of the earth's surface.

2°. From the data thus obtained, to form on paper a representation or map of this surface.

3°. To compute its superficial content or area.

4°. Conversely, to lay out the boundaries of certain portions of the earth's surface from surveys previously made or other data; or to divide a given portion of this surface into smaller portions according to prescribed conditions.

When the portion of the earth's surface is small, the curvature of the earth may be neglected, and the portion surveyed may, without sensible error, be regarded as a plane surface. The work is then called *Plane Surveying.*

The operations in the field or territory to be surveyed consist chiefly in the measurement of certain lines and angles. We commence with a brief description of the instruments employed for this purpose, and of the manner in which they are used.

MEASUREMENT OF LINES.

The Chain and Tape.

98. The instrument in common use for the measurement of lines in Plane Surveying, is Gunter's Chain; so called from the name of its inventor. It is four rods, or 66 feet in length; and is divided into 100 links. Each link is, therefore, 7.92 inches. From the division, moreover, the links, it is evident, will be so many hundredths of the chain. Thus 13 chains 45 links may be written 13.45 chains, or 1345 links.

Part of the first link at each end of the chain is formed into a ring for the purpose of holding it more easily in the hand. And, for greater convenience in counting the links, tallies, consisting of small pieces of brass, are placed at every tenth link from each end of the chain and notched to mark their distance from the end. A round piece of brass is placed at the 50th link to denote the middle of the chain.

In uneven ground a chain four rods in length is found inconvenient, and a chain of two rods, divided into fifty links, is commonly employed. To determine the number of rods, and the decimal parts of a rod contained in a given number of chains and links of this description, we multiply, as it will be easy to see, the chains by two, and the links by four for hundredths. Thus in 7 chains 13 links there are 14.52 rods.

99. To measure a line with a chain two persons are necessary, one at each end of the chain. The one that carries the forward end of the chain is called the *leader*, the other is called the *follower*. At the outset the leader is usually provided with ten arrows or

pins of about a foot in length, as marks to be placed in the ground at the end of each chain, as the measurement proceeds. He is also provided with some suitable contrivance, as a string attached to his person in which a knot may be tied, to serve as a tally at every ten chains.

In order to measure the distance in a straight line between two points A and B, the following process is observed. A stake or signal being placed at the point B, the follower places his end of the chain upon the point A, and directing the leader right or left as the case may require, the chain is drawn out in the direction from A to the signal at B. When it is near being stretched the follower cries *down*, and the leader, drawing the chain tight, puts down a pin. Both then advance with the chain until the follower again cries *down*. The follower then places his end of the chain at the pin left by the leader, and the chain is again carefully stretched in the direction of the signal at B, and the leader puts down another pin. The process is thus continued, if the line exceeds ten chains in length, until, the chain being still stretched in the direction of the signal at B, the leader, having put down the last pin, cries *tally*. At this the follower, the chain being still left upon the ground, advances to the leader and takes up the pin last put down, marking carefully the point upon the ground at which it was placed. He then ties a knot in his string as a tally, and passes the ten pins, now in his hands, into the hands of the leader; and the operation is thus continued until the whole line is measured.

In the measurement of a line the chain must be kept *horizontal*. This will require, if the ground is ascending, that the follower raise his end of the chain until it is precisely on a level with the forward end, taking care, when the marking pin is put down, that his end of the chain is in a plumb line passing through the starting point, or the pin last put down. If the ground is descending, the leader must lift his end of the chain in like manner, observing to put down his pin directly under it, at the point where a plumb line passing through his end of the chain would intersect the ground. In either case, if necessary, the chain should be supported at the middle, so as to be kept perfectly straight.

The chief points to be attended to in chaining are, 1st, to keep the chain without deviation in the direction of the line to be measured. 2d, to keep the chain in a horizontal position. 3d, to record correctly the number of chains. The process above explained is well adapted to fulfill these conditions.

100. In the process of chaining with a two rod chain, the following results were found.

1. The follower, at the end of the process, finds five knots tied upon his string, seven pins in his hand, and that there are 15 links between the point where the last pin was put down and the end of the line measured. What was the length of the line?

Ans. 114.6 rods.

2. The knots or tallies being 12, the odd chains 5, and the links 27½, what was the distance measured?

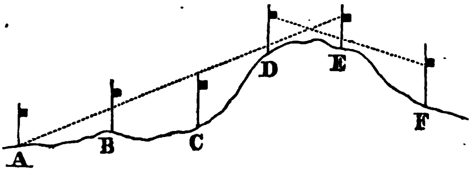
Ans. 251.1 rods.

101. In what precedes we have supposed the line to be measured to be of moderate extent, or to be well defined by natural or artificial marks. In this case it will be sufficient to place a stake or signal at each end of the line. But when the line is of considerable extent, and is not thus defined, it will be necessary to place stakes, at suitable intervals along the line, to serve as a guide to the chainmen.

The stakes should always be driven into the ground perpendicularly to the horizon, so as to be all in the same vertical plane passing through the line. A stake, it is evident, will always be in the line, if made to range with two other stakes already placed upon it. By means of this principle the measurement of a line may be easily extended as far as we please, after two points upon it in sight of each other have been determined.

102. When the extreme points only of the line to be measured are given, and an intervening hill prevents their being seen,

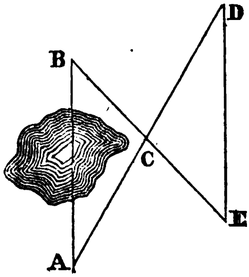
one from the other, two points of the line may readily be determined upon the summit of the hill in the following manner.



Having selected a convenient position upon the summit such that two stakes D and E intended to be placed at two points upon the line can be seen each from the two extremities A and F, the surveyor places the stake D, for example, as near as he can judge upon the line. From D he then aligns the stake E upon F, or places it on the line DF. Next, from E he aligns the stake D upon A, and so on. The stakes D and E are thus brought nearer, at each operation, to the line AF, until they are finally found upon it. As many other stakes B, C, &c. may then be placed upon the line, as its accurate measurement may require. The hill is given in profile, and the final position only of the stakes D and E is noted.

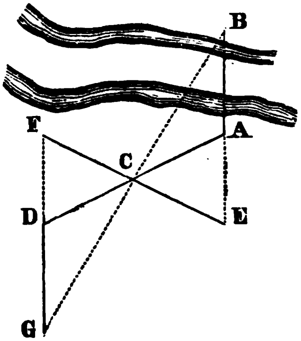
103. It is sometimes inconvenient or impracticable, on account of intervening obstacles, to measure a line directly. Many ingenious contrivances, depending upon the elementary principles of Geometry, are employed to overcome this difficulty. We notice as specimens the two following.

1. Both ends of the line accessible.



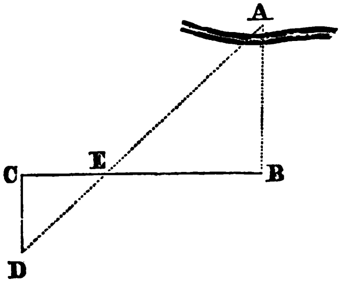
Let AB be the line it is required to measure; but which, on account of an intervening pond or morass cannot be measured directly. Commencing at A we measure an oblique line AD, passing the obstacle at C, and such that CD shall be equal to CA. Placing stakes at D and C, we measure next the line BC which we produce to E making CE equal to BC. Measuring next from E to D we have the length of AB required,

2. One end only of the line accessible.



Let AB be the line it is required to measure, one end only of which, A for example, can be reached on account of an intervening river. We measure first any convenient line AD diverging from the river; and place stakes at the extremity D and middle point C of this line. We take next a point E in the direction BA, and through C we measure the line EF, making CF equal to CE. Placing a stake at F we find by trial a point G such that G will be in the same line with D and F, and with C and B. Measuring lastly GD, we shall have GD equal to AB or the line required.

104. By these contrivances it will be seen, that for the line required to be measured another is substituted equal to it in length; and in a situation in which it can be measured. When this substitution cannot be made, the following process may be employed.



Let BA be the line to be measured, one end only, B, of which can be reached. Lay off a line BC of any convenient length perpendicular to BA, and place a stake at C. From C lay off a line CD of any convenient length perpendicular to CB; and place a stake at D. Find next, by trial, a point E such that E will be at the same time in both the lines CB and DA; and measure the lines BE, EC and CD. Then to find BA we have EC : CD : : BE : BA.

To lay off a perpendicular from C to a given line CB in the preceding figure, the following method may be employed. From C measure on the line CB a distance CE equal to thirty links. Making next one end of the chain fast at E and the ninetieth link at C, take hold of the fiftieth link and stretch the chain, so that the two parts CD and DE will be drawn equally tight. CD will then be perpendicular to CE.

Any of the methods explained in the section on Heights and Distances, may, it is obvious, be employed to determine the distance between two points, one or both of which are inaccessible. The preceding methods require the use of the chain only.

105. In addition to the chain, a tape line divided into feet and inches on one side, and into rods and links on the other, is a convenient instrument for the measurement of lengths. Both, especially the latter, are liable to become inaccurate by being stretched in their use. They should, therefore, be frequently compared with some standard.

MEASURE OF SURFACES.

106. The surface of land is usually estimated in acres, roods and perches, according to the following table,

One square rod = 1 perch
Forty perches = 1 rood
Four roods = 1 acre.

The chain being four rods, a square chain will equal sixteen perches, and we shall have

10 000 square links = 1 square chain,
1 square chain = 16 perches,
10 square chains = 1 acre.

Ex. Let it be required to find the number of acres, roods and and perches in 1878203 square links. The operation will be as follows

$$\begin{array}{r} 18.78203 \\ 4 \\ \hline 3.12812 \\ 40 \\ \hline 5.12480 \end{array}$$

To find the acres we divide the number of square links by 100 000, or which is the same thing, cut off five places to the right for decimals. This gives 18 for the acres. We next multiply the decimal part of the result by 4, cutting off also in the product five decimal places as before, which gives 3 for the roods. Multiplying again, in like manner, by 40 we obtain 5 for the perches. We thus find in the proposed 18 A. 3 R. 5 P.

If the area is expressed in chains, and decimal parts of a chain or links, we divide by 10; or, which is the same thing, cut off one decimal place for acres, and proceed as before. Thus in 36.8679 square chains, we have 3 A. 2 R. 29.9. P.

In the measurement of lines, the chain we have seen is required to be kept horizontal. The area of the surface will be, therefore, what it would be, if the whole were projected upon a horizontal plane. The reason for this is 1°, to introduce uniformity in estimating the area of land. 2°, Because the surface of an inclined field is no more valuable for building or agricultural purposes, than that of its projection on a horizontal plane. Thus on the lines AB, BC, (fig. 33) considered as situated on the inclined sides of a hill, no more trees or grass can grow, than would grow upon the projections of these lines AD, DC, on a horizontal plane.

MEASUREMENT OF ANGLES.

The Surveyor's Compass.

107. The instrument commonly employed in Plane Surveying for the measurement of angles, is the *Surveyor's Compass.* The principal parts of this instrument are a compass box, a magnetic needle, a pair of sight vanes, and a stand for its support.

Upon the plate of the compass box the principal points of the compass are marked by the letters N. S. E. and W. A *Fleur de lis* is placed at the north point to mark this point more distinctly.

The circular ring of the compass box is divided into degrees and half degrees, each way to 90° from the north and south points respectively.

The sight vanes are constructed with a slit in one corresponding to a thread in the other. They are so adjusted that a plane perpendicular to the face of the instrument, and passing through a slit and its corresponding thread, will intersect the north and south line upon the compass plate.

The needle is nicely balanced upon a pivot at the centre of the compass box, upon which, when the instrument is level, it moves with perfect freedom. When left to itself, after a series of vibrations growing shorter and shorter upon either side, it comes finally to a state of rest in a line either due North and South, or in a line making an angle with this, which may be determined for each place where the instrument is used.

The direction in which the needle settles is called the *Magnetic Meridian*, and the angle by which this varies from the true meridian is called the *variation*.

Spirit levels are usually placed upon the instrument to aid in levelling it. Instead of the sight vanes above described, a telescope with cross threads in the axis, is a more perfect arrangement for the sights of the instrument.

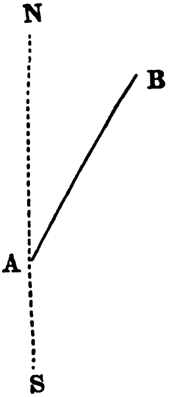
The needle, when left to vibrate freely, will, unless there is some local disturbance, come to a state of rest in lines parallel to each other; and hence the capacity of the instrument for the general purpose of surveying.

108. The angle contained between the direction in which the needle settles and the direction of a given line is called the *bearing* of the line or its *course*.

If the direction of the line from the point from which we start, or from which its direction should be estimated, is between the North and East, the bearing is said to be North so many degrees East; if between the North and West, it is said to be North so many degrees West. If the direction of the line is between the South and East, the bearing of the line is said to be South so many degrees East; if between the South and West, South so many degrees West.

109. Let it now be proposed to find the bearing of a line AB by the compass.

Supposing the point of departure to be the extremity A of the line, for example, and the general direction of the line to be between the North and East, we place the centre of the instrument directly over the point A, and bring it to a level, so that the needle shall have a free and easy motion upon its pivot. The north point on the instrument being from us, we then turn the sight vanes until looking through the slit in the one nearest us, the thread in the one opposite is made exactly to intersect a staff placed vertically at the extremity B, or some other point in the line. The north point on the compass plate will then be on the line, and the distance between this point and that against which the end of the needle settles, estimated in degrees upon the graduated limb, will be the bearing sought.



Ex. The instrument being placed as above, I observed that, when the needle came to a state of rest, it pointed to 40° 30′ on the graduated limb of the compass box. The north point on the compass box being turned from the North toward the East what is the bearing of the line? Ans. N. 40° 30′ E.

If the needle has not come to a state of rest after a series of free and easy vibrations, it should again be put in vibration by a gentle tap upon the instrument.

If after taking the bearing of the line AB from the end A, as above, we again take its bearing back from the other end B, the result is called the *back* or *reverse bearing*.

The forward and reverse bearing, it is evident, are equal angles. The only difference is that they lie between directly opposite points. Thus, if the bearing of AB from A is N. 40° 30′ E. the reverse bearing from B will de S. 40° 30′ W.

The Cross.

110. The cross is another instrument used in surveying. It consists of two sets of sights, similar to those of the compass, placed at right angles to each other; and a stand for its support.

The use of this instrument is to determine perpendiculars to a given line, either from points upon or without it.

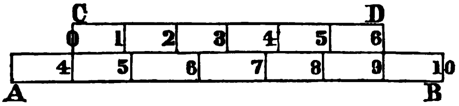
Thus if it be required to determine a perpendicular from a given point upon a line, we place the centre of the instrument directly over the given point, and turning one pair of the sights in the direction of the line, the other sights will mark the direction of the perpendicular to the given point. If the given point is without the line, we place the centre of the instrument upon the line as near as we can judge over the point upon which the perpendicular from the given point would fall, and turn one pair of the sights in the direction of the line. The instrument is then moved to the right or left upon the line, as the case may require, until the other sights are made to intersect the given point. We shall then have the direction of the perpendicular required.

A cross may be easily made in the following manner. Prepare a board about eight inches square. Draw the two diagonals; and at their extremities erect pins to serve as sights. This being placed upon a staff of convenient length, sharpened at the bottom to enter the ground easily, the instrument is prepared for use.

The Theodolite.

111. The only reliable instrument for the accurate measurement of angles, is the *Theodolite*. A complete description of this instrument, with its various adjustments, would exceed our limits. The general view given of the instrument in art. 71, is sufficient for our present purpose. We will merely explain, in general terms, the nature of the contrivance by which the fractional parts of a degree, or any primary division of the limb of the instrument, is effected. This contrivance is called a *Vernier*, from the name of its inventor, and is in principle as follows.

Let AB be any portion of the graduated limb of the instrument, CD a portion of the movable limb to which the telescope or sights is attached. Let CD be equal to any number of divisions on the graduated limb, and let CD be divided into one more parts than the corresponding portion of this limb. Thus let CD be equal to five of the the primary divisions upon AB, and let CD be divided into six equal parts. Then each of the divisions upon CD will be equal to five-sixths of a division upon AB; and the difference between them will be equal to one-sixth of a division upon AB. If then the primary divisions upon the limb AB are degrees, this difference will be one-sixth of a degree or 10 minutes.



Suppose that in the measurement of an angle the 0 on the movable limb, or vernier plate, coincides with 4, for example, on the graduated limb. The index in this case marks 4°. But if the 0 on the vernier has passed a little beyond the 4, we look along the vernier until we find a division on the vernier which coincides with a division upon the limb. Thus, suppose that the division 1 on the vernier coincides with 5 on the limb. The 0 on the vernier must have passed one-sixth of a degree or 10 minutes from the 4; and the index will, therefore, mark 4° 10′. If the division 2 of the vernier is found to coincide with 6 on the limb, the index then marks 4° 20′; and so on.

The vernier is, in general, so adjusted to the graduated limb of the Theodolite, that an angle can be measured to the nearest minute. The same contrivance, it is evident, may be applied to instruments for the measurement of lengths, as well as to those for the measurement of angles.

112. Let ABC (fig. 1) be an angle which it is required to measure with the Theodolite. Placing the centre of the instrument

over the vertex B of the angle, we clamp the graduated limb to the stand, so as to render it immovable. We then turn the sights, or telescope, in the direction of one of the sides of the angle, BA for example, and note the degrees and minutes upon the graduated limb indicated by the vernier. We next turn the telescope in the direction of the other side BC of the angle, and note in like manner the reading of the vernier. If the 0 point of the vernier has not passed the 0 point of the limb, the difference between these two readings will be the angle. But if the 0 point of the vernier has passed that of the limb, we subtract the first reading from 360°, and add the remainder to the second reading.

Ex. 1. The first reading on the limb of the instrument is 54°, 35′; the second 87° 23′; what is the angle? Ans. 32° 48′.

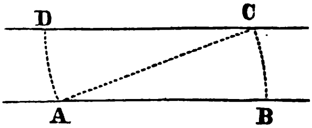
Ex. 2. The first reading is 290° 47′, and the second is 5° 15′; what is the angle? Ans. 74° 28′.

It is usual to place verniers upon opposite sides of the instrument, at the distance of 180° apart. By reading both verniers the angle is measured twice, and upon different parts of the instrument. The mean of the two measurements will be the angle.

INSTRUMENTS FOR PLOTTING A SURVEY.

113. The instruments described above are sufficient for the operations required to be performed in the field or territory to be surveyed. A map or plot of the survey may easily be made by aid of the diagonal scale of equal parts and the line of chords, already described, art. 10 and 13.

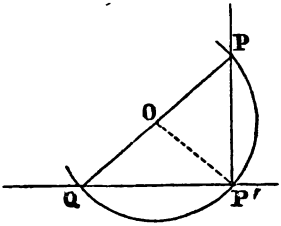
In order tó this there are, in general, two principal operations to be performed, 1°. To draw a line parallel to a given line. 2°. To draw a perpendicular to a given line.



1. Through the point C let it be required to draw a line parallel to a given line AB. Assuming any point A upon the line AB, with an extent equal to AC, describe from A an arc cutting AB in B. Again,

from the point C, and with the same extent in the compasses, describe an indefinite arc AD. Setting off next from A an arc AD equal to BC, we obtain the point D, through which and the given point C drawing a line, it will be parallel to the given line.

2. From a given point P to let fall a perpendicular upon a given line QP′. Upon the given line take any convenient point Q; join PQ and bisect PQ in O. From O, as a centre, with an extent equal to OP, describe an arc cutting QP′ in the points P′ and Q. Join PP′; PP′ will be the perpendicular sought.



3. It will be easy to see how to proceed, if the question is to erect a perpendicular at a given point to a given line.

The Semi-circular Protractor.

114. The nature of this instrument will be obvious. It is simply a semi-circle graduated into degrees and half degrees, and numbered both ways from 0 to 180°.

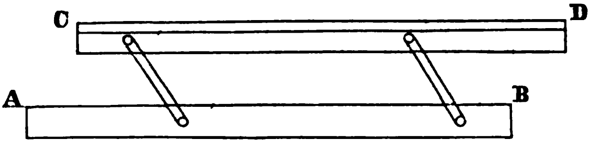
Let it be required to lay off from a point A in a line NS an angle of 40°, for example. We place the edge of the instrument corresponding to the diameter of the semi-circle upon the line NS, and bring the notch, which marks the centre of the semi-circle, upon the point A. Counting the degrees from 0 to 40° on the limb of the instrument, we mark this point upon the paper by a fine prick or dot. Drawing next a line AB through the point A and the prick or dot thus determined, we have the angle sought.

This instrument, it is obvious, is much more convenient than the line of chords for the laying off or measuring of angles upon paper.

The Parallel Ruler.

115. This instrument consists of two flat rulers AB, CD, connected by two cross bars parallel and equal to each other. As the instrument is opened, the rulers will, it is evident, move parallel

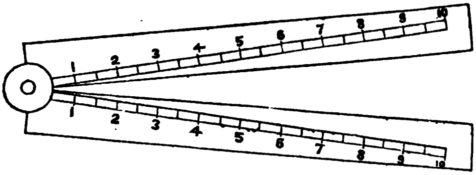
to each other, and it will thus be found convenient for drawing parallel lines.



Ex. Through a given point B let it be required to draw a line parallel to a given line NS. We place an edge of one of the rulers upon the given line NS; holding it firmly in this position, we move the other ruler, until one of its edges comes to touch the given point B. Along this edge we then draw a line. This line will pass, it is evident, through the given point and will be parallel to the given line.

The Sector.

116. This instrument consists of two equal arms movable about a pivot as a centre, like a common jointed rule.



We now refer to two only of the lines upon it. These are drawn diagonally from the centre and are divided each into the same equal parts. Let C be the centre, C 10, C 10 the diagonal lines. Let it now be required to adapt the sector to laying down lines upon a scale of 10 feet to the inch. With an extent equal to one inch in the dividers we place one foot 10 at on one of the arms, and open the sector until the other foot of the dividers just reaches 10 on the other arm. Then, from the nature of similar triangles, the extent from 5 on one of the arms across to the same figure on the other, will be 5 feet on a scale of 10 feet to the inch; and so on for any other figure.

In addition to the above instruments, a *square* for raising perpendiculars will be found very useful.

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SECTION II.

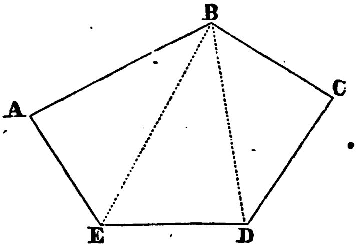
LAND OR FIELD SURVEYING.

117. We are now prepared for all the ordinary kinds of Surveying, such as fields, estates, roads, &c.

SURVEYS MADE WITH THE CHAIN ONLY.

A field may be surveyed with the chain only. In order to this, it will be sufficient to measure the sides of the field and a series of diagonals dividing it into triangles. For greater accuracy in the plotting and calculations, the diagonals should be so selected, that the triangles into which the field is divided may be as nearly equilateral as possible.

Operations in the Field.



Let ABCDE be the field to be surveyed. From a preliminary examination of the ground, it is observed that diagonals may be easily measured from the corner B to the corners E and D of the field, dividing it into the triangles ABE, EBD, DBC, which will fulfill the required condition.

Commencing at B, therefore, we measure in succession the sides BC, CD, DE, EA, AB, of the field, and the diagonals BE, BD. We thus obtain the required data in order to form a plot, and to calculate the area of the field.

118. The minutes of the survey are kept in a book called the *Field Book.* It is, in general, most convenient to commence at the bottom of the page, and read upwards, as follows in the present example.

Sides	Diag.
AB 5. "	
EA 3.20 "	
DE 3.50 "	
CD 3.70 "	BE 5.50 "
BC 3.50 chs.	BD 5.00 chs.

Plotting the Survey.

Beginning at a point B on the paper we lay off from a convenient scale of equal parts a line BC equal to 3.50. Then with the side CD equal to 3.70 and the diagonal BD equal to 5, we construct upon BC the triangle BCD which determines the corners B, C and D of the field. Next, with the side DE and the diagonal BE, we find another corner E; and lastly with the sides EA and AB we find the remaining corner A of the field; and the figure is completed.

Computation of the Area.

119. Having formed a plot of the field we proceed next to compute its area.

By the operations already performed, the field is divided into the three triangles ABE, EBD, DBC, in each of which all the sides have been measured. We shall have, therefore, only to compute the areas of each of these triangles separately, and then to take the sum, which will be the area of the field.

Beginning with the triangle ABE, and applying the rule art. 95, the calculation will be as follows:

$$\begin{array}{lr} \text{Side AB} = & 5.00 \\ \text{" BE} = & 5.50 \\ \text{" EA} = & \underline{3.20} \\ & \underline{13.70} \end{array}$$

Half sum of the sides,	6.85,	log = 0.835691
1st. remainder	1.85,	log = 0.267172
2d. "	1.35,	log = 0.130334
3d. "	3.65,	log = 0.562293
		2) 1.795490
Area of ABE = 7.9022 chs.		= 0.897745

In like manner, the area of the triangle EBD is found to be 8.5732, and that of DBC 6.4707 square chains. Taking the sum of these three areas, we obtain 2 acres, 1 rood, 7.13 perches for the area of the field.

Ex. A survey of a field having been made, it is required to plot the field and find its area from the following notes.

	EA 5.85	
	DE 5.18	
	R. Fr. D.	
	CD 3.65	
	L. Fr. C.	CE 4.20
	BC 3.90	R. Fr. C.
	R. Fr. B.	AC 7.68
Bearing AB, West.	AB 10.54	R. Fr. A.
	Sides	Diag.

To show the position of the field in relation to the points of the compass, the bearing of the side AB the line first measured is taken with the compass. The note R. Fr. A in the field book, denotes that we turn to the right and measure from A; and so of the others.

The plotting of the field will now present no difficulty; and its area will be found to be 3A. 0R. 21.47P.

When a diagonal cannot be measured directly it may be determined by the methods explained art. 103 and 104.

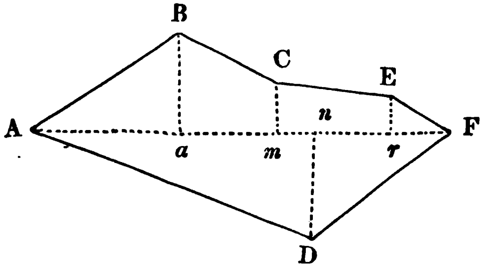
SURVEYS WITH THE CHAIN AND CROSS.

120. It is sometimes more convenient, especially if the sides of the field are numerous and of small extent, instead of measuring

the sides, to measure one or more diagonals and then to determine the corners by offsets or perpendiculars from these.

Operations in the Field.

Suppose the field to be of the form represented in the figure. It will be easily seen that the diagonal from A to F will be best adapted to our purpose.



Beginning, therefore, at A we carefully mark by stakes the diagonal AF. We next determine the points a, m, n, r, where the perpendiculars from the remaining corners B, C, D, E, will fall upon this diagonal; and then measure the distances Aa, Am, An, Ar to the feet of the perpendiculars Ba, Cm, Dn, Er, as also the perpendiculars themselves, and the whole diagonal AF. The field is thus sufficiently determined.

The perpendiculars may be found by means of the chain only by the methods explained, art. 104. They may, however, be determined with far greater facility by aid of the cross which we suppose to be employed for the purpose. The mode of using it has already been explained.

The following is the field book in the present example. The perpendiculars or offsets to the right are placed in the right hand column; and those to the left in the left hand column. The bearing of the diagonal is taken with the compass.

	AF	
	18.96	Diag.
E 2.59	13.42	
	11.32	3.25 D
C 3.47	7.63	
B 7.56	5.22	
Begin	at A	Range East.
Offsets L.	Diag.	Offsets R

Plotting the Survey.

To plot this field, we lay off first a line AF equal to 18.96. Then from A we lay off upon this line a distance A*a* equal to 5.22; and from the point *a*, toward the left erect the perpendicular *a*B equal to 7.56. We thus obtain the corner B; and so on for the remaining corners of the field.

Computation of the area.

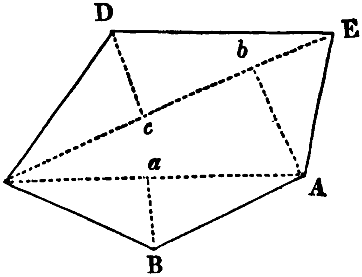
The field is already divided by the measurement into the triangles AB*a*, *r*EF, FDA, and the trapezoids *a*BC*m*, *m*CE*r*. To find the area of the field we have, therefore, only to find the areas of these several figures and to take their sum. The area of a trapezoid, it will be recollected, is equal to the product of half the sum of the parallel sides by the altitude. The area of the trapezoids will then be readily found, since we have given the two parallel sides in each, together with the altitude. And to find the area of the triangles, we have given the base of each and the altitude. The double areas of each of the figures, will then be found as follows,

Triangle AB*a*	= 39.4632 chs.
Trapezoid *a*BC*m*	= 27.0643
Trapezoid *m*CE*r*	= 36.2454
Triangle *r*EF	= 14.3486
Triangle FDA	= 61.6200
	178.7415
Whole area	= 8.93707 chs = 8A. 3R, 29.9 P.

121. The perpendicular B*a* which measures the distance of the point B from the diagonal AF is called the *ordinate* of this point; and when a point B is determined by two lines A*a*, *a*B these lines are called *co-ordinates* of the point.

Ex. 1. Let it be required to plot a field and find its area from the following field notes.

	CE	
	16.66	Diag.
	13.46	4.96 A.
D 3.76	5.73	
	R Fr. C	
	AC	
	14.33	Diag.
B 2.73	6.43	
Begin	at A	Go W.
	Diag.	



From the field book it appears that two diagonals were measured in the survey. We begin by drawing a line AC, from A toward the left, equal to 14.33, the first diagonal. Setting off on this line $Aa = 6.43$, from a we raise, toward the left, a perpendicular $aB = 2.73$; and join AB, BC; we thus have two sides of the field.

The note R. Fr. C in the field book denotes, as we have seen, that we turn to the right and measure from C. To draw the diagonal CE, therefore, we take in the compasses $4.96 = Ab$, the perpendicular from A upon this diagonal; and, with this extent, as radius, from A describe an arc; next with the extent 13.46, the distance from C to the foot of the perpendicular Ab, we describe another arc cutting the former in b; then through b we draw the diagonal $CE = 16.66$. Finally we lay off on CE the distance $Cc = 5.73$, and having drawn, to the left, the perpendicular $cD = 3.76$, we join CD, DE, and thus obtain the plot required.

Calculating the areas of the several triangles into which the field is divided and taking their sum, we obtain 9 A. 0 R. 35 P. for the area of the field.

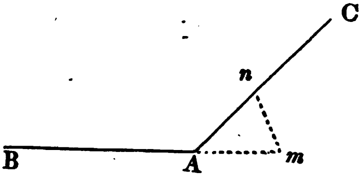
Ex. 2. To plot a field from the following notes.

	CH	
	16.35	Diag.
$*_1$	9.10	3.90 I.
D 2.35	5.75	
	R Fr. C	
	AC	
	11.65	Diag.
B 2.63	5.30	
	4.00	6.30 I.
Begin	at A	Go S. 67° E.

	EG	
	11.50	Diag.
F 4.00	7.17	
	R Fr. E	
	HE	Diag.
	14.74	
D 4.65	9.75	$*_2$
$*_3$	4.65	5.75 G
	L Fr. H	

The direction of the diagonal CH is determined from the points C and I by the co-ordinate lines 9.10 and 3.90; the direction of HE from the points H and D by the co-ordinates 9.75 and 4.65; and EG by the co-ordinates 4.65 and 5.75. For distinction the co-ordinate lines employed in determining the direction of the diagonals are marked with a star in the field book, with small subscript figures denoting the order in which they are employed.

122. If the diagonals cannot be measured directly we may employ any of the methods explained art. 103, 104, for obtaining their lengths. If the measurement becomes very inconvenient or impracticable, we then measure the sides of the field and determine their inclination to each other in the following manner.



Let BA, AC be two sides of the field. The angle BAC being obtuse, we produce BA any convenient distance A*m*; we then measure on AC an equal distance A*n*, and measure *mn*. The angle BAC will be determined, it is evident, by this measurement; since *mn* will be the chord of its supplement, the radius being A*m*.

Let $Am = 1$ ch, and $mn = 87.67$ links. To plot the angle, we take on the line BA, produced as far as is necessary, a part $Am =$ 1 ch; and with this as a radius describe, from the point A, an indefinite arc mn. Applying next to this arc, from m to n, a chord equal to 87.67 links, and drawing AC through the point n, thus determined, we have BAC the angle required.

The chord mn, it is easy to see, is equal to twice the sine of half the angle mAn; or half the chord mn is equal to the sine of half the angle mAn. We shall have, then, 43835 for the sine of half mAn; which corresponds in the table of natural sines to 26° very nearly. Thus $mAC = 52°$; and by consequence, $BAC = 128°$.

If the angle BAC were acute, we should then measure Am upon AB to the left of A. The field book will always indicate the manner in which the measurements are made.

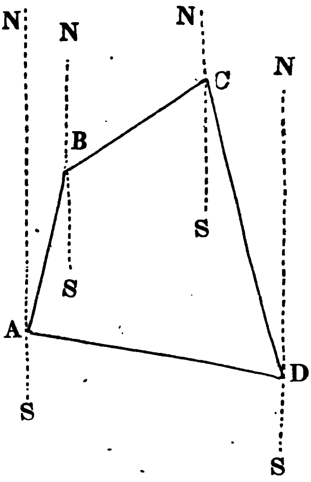
Ex. To plot a survey from the following notes

	DA 4.76	
	L Fr. D	
	CD 6.18	
	L Fr. C	
	BC 4.43	
	L Fr. B	
	AB 7.23	
	.96	Chord.
	1.00	On the sides BA, BC.
Begin	at A	Range E.

The area of the field may be computed in the manner explained in the following article.

SURVEYS WITH THE CHAIN AND COMPASS.

123. The preceding method is eminently adapted to level and open fields; since the lines in this case may easily be measured with great accuracy. But if the ground is very uneven, or is thickly covered with wood, as is usually the case in new countries,



the method becomes inconvenient, or impracticable; and the Surveyor's Compass is introduced, in connection with the chain, to make the measurements required for the determination of the field.

Let the field be of the form represented in the figure; the upper side being regarded as North, the right hand East, and so on.

Operations in the Field.

Beginning at some convenient corner of the field, A for example we take the bearing of the side AB with the compass, and then measure its length with the chain. Removing next the compass to B, we take the bearing of the side BC and measure its length, as before. And so on, until, passing round the field we return to the point A from which we started. The several bearings and distances may be noted in the field book in the following manner. In the first column we write down the number of the station; in the second the bearings; in the third the distances; and in the fourth, headed remarks, we enter anything deserving special notice in the course of the work. Thus, in the present example, we have the following field book.

Stations.	Bearings.	Distances.	Remarks.
4	N. 77° W.	23.66 "	The tower bears N. 50° W.
3	S. 14° E.	23.00 "	
2	N 83° E.	11.00 "	
1	N 23° E.	17.00 chs.	A Tower bears N. 45° E.

In the column of remarks it will be observed that the bearings of a tower situated in the field are taken at the stations 1 and 4.

Plotting the Survey.

124. To form a plot of the survey, we assume a point A upon the paper as the first station point, and through this point draw a dotted line NS to represent a meridian line. We then lay off on the right of the line NS an angle NAB equal to the given course 23°. Then through this angle we draw the line AB equal to the distance 17 chains. By aid of the parallel ruler, we draw next, through the point B, another dotted line NS to represent the meridian through B; we then lay off to the right the angle NBC equal to the next course 83°; and through this angle draw BC equal to the next distance 11 chains; and so on for the remaining sides of the field.

If the measurements are all accurately made in the field, and the work is also correctly plotted, on laying off the last line on the required course and of the given length, the end of this line will fall exactly upon the point A, at which we began.

When this is not the case, some error must have occurred, either in the survey or the plotting. If not found in the latter, it should be sought in the field. The lines and angles being first re-measured, in which it is most probable the error has occurred.

The reverse bearing may be taken, as we proceed, as a test of the accuracy of the direct bearing.

In the present example it remains only to mark on the plot the place of the tower referred to in the remarks. In order to this, at A we lay off, to the right, an angle equal to the bearing, 45°, of the tower at that station, and through this angle draw an indefinite line. We next at the station D lay off, to the left, an angle equal to the bearing, 50°, of the tower from that station, and through this angle draw an indefinite line intersecting the former. The point of intersection of the two lines will be the place of the tower on the plot.

Computation of the Area.

125. To find the area of the field we divide it into two triangles by a diagonal BD drawn from B to D. We shall then have given in each of the triangles ABD, BCD two contiguous sides and the included angle, from which, art. 94, the area of the triangles will be easily found.

Indeed, in the triangle ABD the sides AB, AD are given directly from the measurements; and since the angle NDA is equal to its alternate SAD, we shall have for BAD the angle included between these sides, $BAD = 180° - (NAB + NDA)$; that is, to find the angle BAD we subtract the sum of the bearings of the two sides AB, AD from 180°. We have, therefore, for the double area of the triangle ABD

$$\begin{aligned} 2A &= AB \times AD \sin\left[180° - (NAB + NDA)\right] \\ &= AB \times AD \sin 80°; \end{aligned}$$

or, performing the operations

$$\begin{array}{rcl} \log 17 & = & 1.230449 \\ \log 23.66 & = & 1.374015 \\ \sin 80° & = & \underline{9.993351} \\ 2A = 396.11 & = & 2.597815 \end{array}$$

Again, in the triangle BCD we have given directly the two sides BC, CD; and to find the included angle, we have $BCD = NBC + SCD$: that is; to find BCD we take the sum of the bearings of the two sides BC, CD; whence

$$2A = BC \times CD \sin (NBC + SCD) = BC \times CD \sin 97°.$$

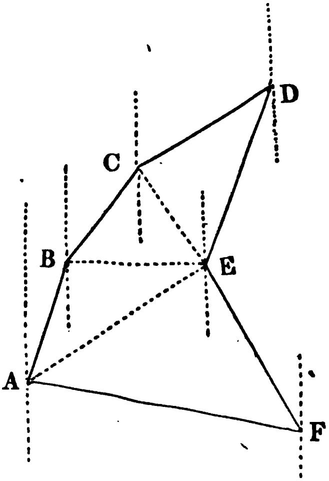
Performing the calculations we obtain $2A = 251.13$ square chains. Adding together these two areas and taking half the sum, we obtain finally for the area of the field 32.362 acres; or 32 A. 1 R. 18 P.

126. In the survey of a field the following notes were made, viz.

Stations.	Bearings.	Distances.
6	N. 72° W.	12.25 "
5	S. 6° 20′ E.	9.60 "
4	S. 21° W.	8.50 "
3	N. 68° E.	7.50 "
2	N. 41° E.	7.62 "
1	N. 18° E.	5.40 chs.

It is required to plot the field and find its area.

The plot of the field will be as in the figure. To find the area we draw diagonals dividing the field into the triangles CDE, BCE, ABE, AFE.



Commencing next with the triangle CDE, we have, from the measurements in the field, the sides DC, DE, and the angle CDE contained between them; whence, putting A for the area, we have

$$2A = DC \times DE \times \sin CDE.$$

We next calculate in the triangle CDE, the remaining angles DCE, CED, and the the third side CE; then in the triangle BCE, we have the sides BC, CE; and subtracting the computed angle DCE from BCD, given by the measurement, we have also the angle BCE contained between these sides. Whence in the triangle BCE we have

$$2A = BC \times CE \times \sin BCE.$$

We proceed in like manner to find for the triangle ABE

$$2A = AB \times BE \times \sin ABE.$$

Lastly, in the triangle AFE, we have

$$2A = AF \times FE \times \sin AFE.$$

Adding these several areas together, and taking one half the sum, we shall have the area of the field.

Performing the calculations, which now present no difficulty, we have for the area sought, 12 A. 1 R. 5 P. Ans.

Ex. Plot a survey and find the area from the following field book.

Stations.	Bearings.	Distances.
6	N. 8° 30′ W.	52 "
5	S. 54° W.	55 "
4	S. 34° 45′ E.	32 "
3	S. 26° W.	35 "
2	N. 70° 30′ E.	40 "
1	N. 25° E.	29 chs.

Ex. 2. To plot a survey from the following notes.

Stations.	Bearings.	Distances.
7	N. 45° 30′ W.	27 "
6	N. 28° E.	51 "
5	N. 70° W.	72 "
4	S. 25° 30′ E.	49 "
3	S. 30° W.	39 "
2	S. 42° E.	36 "
1	N. 65° E.	40 chs.

In the margin of this field book the following notes were made.

From station 1 to 2 the field lies upon a road, which runs in the direction of this side of the field.

From 2 the spire of a church bears N. 68° E; and from 3 the same spire bears N. 15° E.

From 2 also a tree on the westerly side of another road, intersecting the former by the church, bears S. 72° E; and from 3 the same tree bears N. 70° E.

From 3 another tree by the same side of the road bears S. 41° E; and the same tree from 4 bears N. 70° E.

The corner 5 is on the road. From this corner the road runs by a stone mill which bears S. 44° W. At 6 the mill bears S. 25° E. This corner touches a small stream which turns the mill.

Let the learner from these data trace the roads &c. in connection with the plot of the field.

127. The angles which the sides of a field make with each other

may be found from the bearings; and by means of these the field may be plotted without the necessity of drawing meridian lines though the corners.

The angles may be found by the following rules.

1°. If the bearing of one of the sides is north and the other south, one east and the other west, *subtract the less course from the greater.*

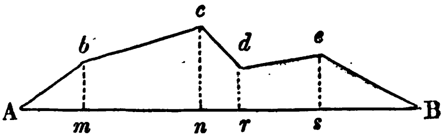
2°. If one is north and the other south, and both east or west, *add both courses together.*

3°. If both are north or south, and one east and the other west, *subtract their sum from* 180°.

4°. If both are north or south, and both east or west, *add the less course to the supplement of the greater.*

The reason for these rules may be shown from the example, art. 125. In general it will be best to make first a rough plan of the field, and then determine the angles by the method which in each case will be thus readily suggested.

128. A field is sometimes bounded in part by lines of small extent. Instead of taking the bearing of these small lines and measuring them severally, it is, in general, best to run a base, or chain line as it may be designated, in some convenient direction as near as possible to the given boundary; and then determine its principal points by means of offsets.



Suppose from A to B the field is bounded by a broken line *AbcdeB*. From A to B we measure a chain line AB, and take its bearing; the principal points *b*, *c*, *d*, *e*, are then determined by the offsets *mb*, *nc*, *rd*, *se*, in the manner already explained.

The area of this offset, it is evident, may be easily found, by calculating the area of the triangles and trapezoids into which it is divided by the operations, and taking their sum.

Ex. To plot a field from the following notes.

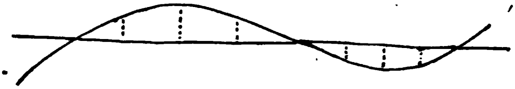
Sta.	Bearing.	Distance.
5	N. 31° W.	9.40
4	S. 70° W.	10.90
3	S. 20° W.	5.30
2	S. 60° E.	11.85 C.L.
1	N. 45° E.	9.30

.00	11.85	C. L. from 2 to 3.
1.50	10.00	
6.00	8.50	
2.00	7.00	
.75	5.20	
2.25	4.00	
1.75	2.20	
.00	.00	

Chain Line.

To find the area of this field, we first calculate the area of the figure included by the chain line and the other principal bounding lines; we then calculate the area of the offset from the chain line, as explained above; and the sum of these two areas will be the area of the field.

129. Instead of a broken line, a field is sometimes bounded by a line irregularly curved, as by the margin of a brook, river or lake. In this case we run, as before, a chain line as near the boundary as possible, and by means of offsets determine a sufficient number of points in the curve to draw it to the degree of accuracy required.



Ex. Plot a curve line offset and find its area from the following notes.

AB	
10.12	.00
8.92	.53
7.86	.80
6.45	.95
5.00	.45
3.50	.63
2.00	.84
1.00	.52
0.00	.00
Begin at A	Range W.

The area is 2 roods, 18 perches.

130. In general, if the boundaries of a field are difficult of access, or are very irregular, we run chain lines as near them as is convenient, and then determine the prominent points by means of offsets.

Ex. 1. A piece of ground adjacent to a river having been surveyed, it is required to make a plot, and show how its area may be found from the following notes.

Chain Lines.

AD	S. 88°	E.	24.63 chs.
CE	N. 48°	W.	10.60.
EB	S. 61° 30′	W.	13.50.

Offsets from the chain lines.

		CE		
		10.60	0	
		10.00	.50	
		9.00	1.10	
		8.00	1.54	River.
		7.00	1.73	
		6.00	1.42	
		4.00	0	
		2.00	0	
		.70	82 *m*	
	Return	to C,	go to E.	
		AD		
	.00	24.63		
	.70	23.80		
River.	*m* 1.10	22.00	C	
	E 6.98	14.10	1.50 F.	
	a 1.65	2.50	B	
	.00	0.00		
	Begin	at A	go to D.	

EB		
13.50	0.	
12.65	1.40 *a*	
12.00	1.70	
11.00	2.44	
10.00	2.50	
9.00	1.90	River.
8.00	1.00	
7.00	0	
6.00	0	
5.00	.94	
4.00	1.42	
3.00	1.53	
2.00	.70	
0.00	0	
L. Fr. E.	go to B.	

The chain lines BE, CE, proceed from the points B and C on the chain line AD; the former at the distance 2.50, the latter at 22.00 from A, as noted in the field book. Offsets to the point *m* are made both from the chain lines AD and CE; and to the point *a* both from AD and EB.

A road three rods in width passes along the side AF of the field. At the point F it turns in the direction S. 67° E. Near the side of the road opposite the corner F of the field, is a dwelling house with other buildings.

At the station A, a mark near the water's edge, on the opposite side of the river bears N., and at B N. 51° W. The river is known to be 1 chain, or 4 rods, wide at B, and 3 rods wide at D.

From D a house, situated on the bank of the river and by the side of the road from F, bears S. 15° E.

Ex. 2. It is required to draw a plan of a field from the following notes in the field book.

Stations.	Bearings.	Distances.
E	N. 45° W.	8.70 C. L.
D	S. 66° W.	4.00
C	S. 17° W.	6.90
B	S. 50° E.	8.40
A	N. 35°15′ E.	9.20 C. L.

Offsets from the chain lines.

	AB	
	9.20	.0
	8.00	.70
	7.50	.30
	6.00	1.00
	5.00	.50
.70	3.30	
.40	2.00	
.60	1.20	
.00	0.00	
Begin	at A	go to B.

	EA	
.00	8.70	
1.00	7.20	
	5.50	.80
	4.00	1.50
	2.50	.50
	1.40	.60
	0.00	.0
Begin	at E	go to A.

From the field book it appears that the chain lines lie partly within and partly without the field. In estimating the area of the field, the offsets connected with chain lines, or parts of them, without the field, will be, it is evident, subtractive.

Ex. 3. To plot a field from the following notes.

Sta.	Bearing.	Distance.	Remarks.
5	S. 72° W.	5 70 C. L.	From 5 to 1 bounded by a broken line.
4	S. 33° E.	3 00 C. L.	
3	S. 35° W.	3 98 C. L.	From 3 to 4 and 5 bounded by a creek.
2	N. 71° E.	4.67	
1	N. 15° E.	6.30 C. L.	Bounded on this side by a curved road.

Offsets from the chain lines.

.00	6.30
.50	4.60
.40	4.00
.20	3.10
.35	2.10
.60	1.30
.30	.70
.00	.00

1 to 2

.24	3.70	
.00	3.00	
.30	2.20	
.40	1.60	
.25	1.10	
	.60	.22
.00	.00	

3 to 4

.00	3.00
.80	2.30
.90	1.75
.50	1.00
.23	.00

4 to 5

.00	5.70
.25	4.40
.70	3.60
.40	2.80
1.00	2·20
.00	.00

5 to 1

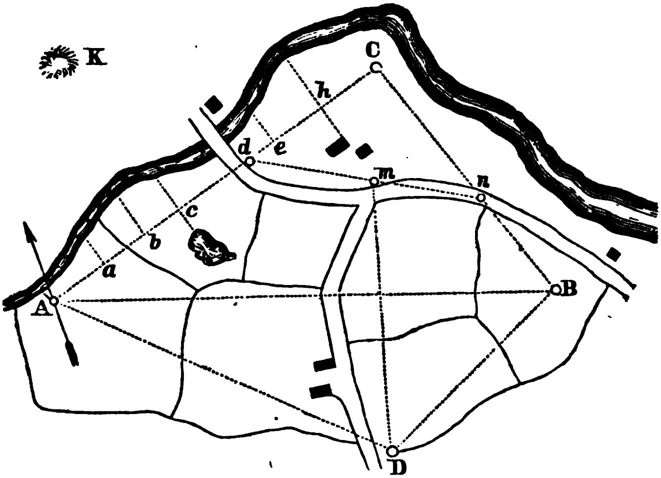
Proof Lines.

131. If in plotting a field the line last laid down exactly closes the field, it may be inferred, as we have seen, that the operations in the field and the plotting have been correctly performed. In addition to this general test, others may be introduced. Thus in example 1 of the preceding art. the perpendicular 6.98 to E at the distance 14.10 from A, is not necessary to the plotting, but serves as a proof line to show whether the point E is correctly determined by the other data. In the figure, art. 121, the diagonal DA may be measured as a proof line. In general, if a field is bounded by a large number of sides, it will be best to establish several proof lines; so that if errors in the operations have occurred, the part of the field in which they were made, may be more readily determined.

SURVEY OF A LARGER TRACT OF LAND, OR ESTATE.

132. Thus far we have confined ourselves to the survey of a single field, noticing such objects within or contiguous to it, as appeared of particular interest. We proceed next to the survey of lands still of moderate extent, but embracing several fields or in-

closures. Let the following figure represent the estate to be surveyed.



The surveyor first walks over the ground, noticing the prominent objects to be determined, as roads, buildings, fences, &c.; and placing signals at the points best adapted to form the principal stations of the survey.

In the figure let it be supposed that the points A, B, and C can be seen from each other, as also the points A, B, and D; and that nothing prevents the measurement of the lines AB, BC, &c. which join these points. Signals are placed at the points A, B, C, and D, to mark the primary stations of the survey. The work is commenced with the measurement of the line AC, for example, from which offsets are made to the river, or other prominent objects situated near to this line. At *d* near the road a signal is left for a secondary station in filling up the survey. Measuring next the line CB, offsets are again made to the river and other prominent objects near this line; and a signal is left at *n* on this line to mark another secondary station. In like manner the remaining lines joining the primary stations, are measured; and offsets to the prominent objects near them determined. Signals are also left at such points upon them as will best serve for secondary stations in filling in the work.

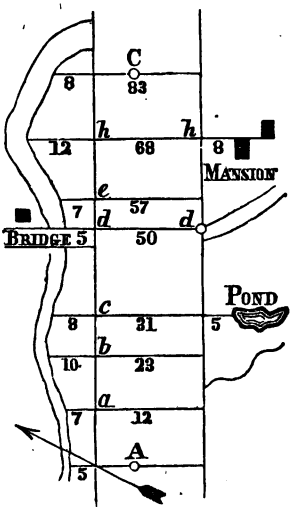
The work connected with the primary lines of the survey being thus performed, the surveyor proceeds next to measure the lines which join the secondary stations, making offsets from these also to such objects near them, as it may be required to notice in the survey. Returning to *d*, for example, the line *dn* joining the stations *d* and *n* is measured, a signal being left at *m* for a new secondary station; and so on. The work is thus carried on, until all the details are filled in, and the whole is completed.

133. The field book may be kept in the following manner.

Each page is divided into three parts by vertical lines. The middle one is the smaller, and the other two are made equal to each other. The notes begin at the bottom of the page, and read upwards. In the middle column are entered the measurements upon the principal lines of the survey; and in the right and left hand columns the offsets to the principal objects, which are sketched as the work proceeds.

Taking the line AC, for example, of the survey, the measurements on this line beginning at A, are entered in the middle column, and the offsets from this line, in the right and left hand columns. A 0 is placed at A to indicate the station.

Thus at *c*, the distance A*c* from A, equal to 31, is written beneath a horizontal line; and beneath the same line the offset 5 to the pond on the right, and 8 to the river on the left. A sketch is made of the river in the left hand column, and of the pond, mansion-house, &c., in the right, as nearly in their exact proportions and relations as the eye can judge.



In like manner all the details of the survey are entered in the field book; and a map or rough sketch of the different objects made as the work proceeds. This map will be found of material service in plotting the survey. Details of minor importance may also be determined by the rough sketch with sufficient accuracy. Any minute sinuosity of the river, for example, between the offsets may be laid down in this manner.

At the station A the bearing of the line AC, for example, may be taken, in order to indicate the position of the estate in relation to the points of the compass. In the field book the direction is noted by the arrow; and the bearing should be entered at A in the middle column. Care should also be taken to measure on different parts of the estate, such proof lines as may be necessary to test the accuracy of the work.

A plot of the survey having been made, the area of the whole, as also of the several inclosures, if required, may be computed in the manner already explained.

134. In the present example the principal lines, and all the work necessary to make the plot, have been determined by the chain only. This cannot always be done with convenience, and an instrument for the measurement of angles will become necessary.

Supposing the position of the principal stations accurately determined in the manner already explained, the compass may be employed with advantage in filling in the details of the survey. Beginning at A, for example, as many courses as we please may be run along the margin of the river, and offsets be made to it from each of the courses. A minute trace of the shore line may thus be made from the field book. Commencing at A the work should terminate at the next principal station C; and beginning again at C it should terminate at B, and so on. Thus the errors which have occurred between A and C will not be propagated beyond C so as to affect the work between C and B. To avoid as far as practicable the errors of the compass, the chain and compass work should always be thus connected with the principal stations of the survey.

By means of the compass we may always, as in preceding examples, determine the position of objects, either within or without the tract surveyed, by taking their bearings from any two stations of the survey. Thus, let it be required to mark on the plot the position of the hill K. This may be done by taking its bearings either at the stations A and C, or at A and B. And so for any other objects we may wish to place upon the map.

Ex. In the survey of an estate the following notes were made; it is required to plot the survey.

The estate is bounded as follows; on the north by a county road; east by a stone wall; south by a river; west by a small stream or brook, which empties into the river.

Primary Stations.

Four primary stations were selected in the survey, the lines joining which nearly inclose the estate. Two of these stations C and D are on the county road, and the other two on the margin of the river.

	AB	
	31.00	26.60 to station C " "
	0.00 chs.	24.00 chs. to station D on the county road.
Begin	at A	Course W. to B.

Offsets from the lines joining the primary stations.

1. From AD to the stone fence.

	AD	
	23.50	1.50
	15.00	4.00
	11.00	7.00
	8.00	5.00
	0.00	1.00
Begin	at A	go to D.

2. From AB to the river.

	AB	
.20	31	
3.00	25.30	
4.00	22.40	
2.20	21.00	
.20	17.60	
1.60	14.40	
2.40	11.00	
2.10	6.00	
.20	1.00	
.80	0.00	
Begin	at A	Range W.

River 4 chs. wide at A.
4.50 " at B.

3. From BC to the brook.

	BC	
.90	26.60	
1.30	25.00	
	21.50	.00
	17.00	1.00
	15.20	2.00
.00	13.50	
3.20	12.40	
1.60	8.60	
.80	6.40	
1.60	3.20	
1.00	0.00	
Begin	at B	

Brook .70 chs. wide at B;
.30 " " at C.

Secondary Stations.

1. A secondary station *a* is taken on the southerly line CD of the county road, at the distance 13.80 from C. From this station courses are run along the westerly line of a road leading to the river, in order to determine the road and the adjacent objects.

	d	River.
	10.00	.75
.15	6.00	
.55	2.00	A fence runs W. to the brook.
	S. 2° W.	
	c	Station.
	8.35	.20
	2.00	.50
	0.00	.00
	S. 36° E.	
	b	Station.
	10.20	.00; 2.00 to the mansion-house.
	8.00	.20 A fence runs W. to the brook.
	6.00	.00; 1.00 to a cottage.
	.00	.00 Road three rods wide.
	S. 9° 30′ W.	
Begin	at *a*	

2. Station *b*. From this station courses are run on the northerly line of a road, two rods wide, leading by the farm house.

	D	Primary Station.
	10.90	
	5.00	2.30 to school house.
	m	Station.
	11.60	
Farm house .80	10.00	
	N. 85° E.	
Begin	at *b*	

3. Station *m*. From this station a road, two rods in width, runs S. 60 E. through the estate. Southerly from *m* is the corner *n* of a fence, distant 6.50 from *m* and 5.75 from *c*. The fence runs from *n* to the two roads in the directions *nm*, *nc* respectively. From the corner *n* a fence runs also to station A and the stone wall.

Objects determined by their bearings.

1. From station C a mill, situated on the brook, bears N. 10° E; and from *a* it bears N. 54° W.

2. From station *a* a church, situated by the road, passing through the estate, bears N. 19° E.; and from D it bears N. 52° W.

3. From B a tree on the summit of a high hill bears N. 40° W; and from C S. 35° W.

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SECTION III.

AREAS. RECTANGULAR METHOD OF COMPUTING AREAS AND PLOTTING.

Areas.

135. The elementary forms to which the computation of areas may be reduced, are, as we have seen, as follows:

1. The triangle, the area of which is equal to *the base multiplied by half the altitude.*

2. The rectangle, the area of which is equal to *the product of two contiguous sides.*

3. The parallelogram, the area of which is equal to *the product of two contiguous sides by the sine of the included angle.*

4. The trapezoid, the area of which is equal to *one half the sum of the parallel sides by the altitude of the trapezoid.*

Any figure bounded by right lines may always be reduced to one or more of these elementary forms, and thus its area may be determined.

136. A figure may be bounded in whole or in part by a regular curve. In this case the area will be found by the rule determined for the curve. Thus

1. Let the bounding curve be a circle. To find its area, *multiply the square of the radius by* 3.1416.

2. Let the bounding curve be an ellipse. To find its area; *multiply the product of the semi-axes by* 3.1416.

Ex. 1. A gentleman has in his garden a fish pond the diameter of which is 10 rods. Required the area of the pond.

Ans. 0 A. 1 R. 38 P.

Ex. 2. An island is found to be in the form of an ellipse. The longer axis is 16.08 chs., and the shorter is 9.72 chs. What is the area? Ans. 12 A. 1 R. 4 P.

Ex. 3. A field is in the form of a sector of a circle having 22.50 chs. for its radius, and 23° for its arc. What is the area?

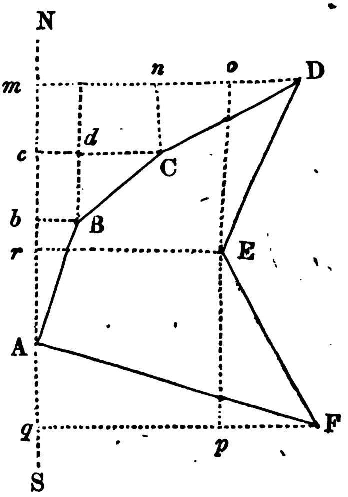
Since the area of a sector is to that of the whole circle, as the number of degrees in the arc of the sector to 360°, we shall have, in this case;

$$A = \frac{(22.50)^2 \times 3.1416 \times 23^\circ}{360^\circ} = 10 \text{ A. } 0 \text{ R. } 25.6 \text{ P. Ans.}$$

Rectangular method of computing areas.

137. The area of a rectilineal figure may, in general, be conveniently found by a process called the rectangular method which we will now explain.

Let the plot of the field be as in the figure. Through the corners A, B, C, &c. of the field draw meridian lines; and assuming one of these, that which passes through A for example, as the *first* or *principal meridian*, let fall upon it perpendiculars from the corners B, C, D, &c. These perpendiculars, it is evident, will be *east* and *west* lines.



The perpendicular distance between the east and west lines, drawn through the extremities of a course, is called the *northing* or *southing* of the course, according as its direction is north or south. This distance corresponds, it is evident, to the difference of latitude between the extreme points of the course; or to the latitude of one of the points reckoned from the other. It is affected with the sign + or —, according as the direction of the course is towards the north or south.

The perpendicular distance between the meridians passing through the extremities of a course, is called the *easting* or *westing* of the course, according as its direction is east or west. This distance corresponds to departure, and should be affected with the sign + or —, according as the direction of the course is towards the east or west.

The perpendicular distance of a point from the first meridian is called its *meridian distance.* It is affected with the sign + or —, according as it lies towards the east or west of the first meridian.

The meridian distance of a line is the distance of its middle point from the first meridian. It is affected, in like manner as the point, with the sign + or —; and is equal to half the algebraic sum of the meridian distances of the extreme points of the line.

138. These things being premised, we commence with calculating the northing or southing, and the meridian distance of each of the corners of the field.

In the triangle A*b*B, we have given the angle *b*AB, and the side AB, from which the northing A*b*, and the easting *b*B of the corner B of the field may be found. In like manner we have given in the triangle B*d*C the angle *d*BC and the side BC to find the northing B*d* and the easting *d*C of the corner C; and so for the remaining corners of the field.

The meridian distance of the corner B is the same with its easting *b*B. The meridian distance of the corner C is equal to its easting *d*C added to the meridian distance *b*B of the preceding corner B; the meridian distance of D is equal to its easting *n*D added to the preceding meridian distance *c*C; that of E is found by subtracting its westing D*o* from the preceding meridian distance *m*D; and so on for the remaining corners of the field.

139. These calculations being performed, to determine the area of the field, we compute first the area of the triangle A*b*B, for which we have the base A*b* and perpendicular *b*B. We next compute the area of the trapezoid *b*B*c*C, for which we have the two parallel sides *b*B, *c*C, and the perpendicular B*d*. Finally we compute the areas of the trapezoids *c*C*md*, *r*E*m*D, *r*E*q*F, and the triangle FA*q*, for which, it is easy to see, we have the requisite data.

If the figure is now examined with attention, it will be seen that a part of the areas we have computed are connected with lines running north, and the rest with lines running south. The former are called *North areas*, the latter *South areas*. Thus the trapezoid *b*B*c*C is a north area; and *rm*DE is a south area. It will be seen, moreover, that the north areas lie entirely without the field; and that the south areas comprehend the field and the north areas also. Hence, if *from the sum of the south areas, we subtract the sum of the north areas, the remainder will be the area of the field.*

For the sake of convenience we have taken the most westerly corner of the field for that through which to draw the first meridian. Any other corner will answer; but, for obvious reasons, it will be best to take either the most westerly or the most easterly corner.

In the present example the data and calculations may be conveniently arranged as in the following table.

Sta.	Courses.	Dist.	N.	S.	E.	W.	M.D.	D.M.D.	N. Areas.	S. Areas.
1.	N. 18° E.	5.40	5.14		1.67		1.67	1.67	.8584	
2.	N. 41° E.	7.62	5.75		5.00		6.67	8.34	4.8530	
3.	N. 68° E.	7.50	2.81		6.95		13.62	20.29	5.7015	
4.	S. 21° W.	8.51		7.95		3.05	10.57	24.19		19.2310
5.	S 6° 18′ E.	9.60		9.54	1.05		11.62	22.19		21.1693
6.	N. 72° W.	12.25	3.79			11.65		11.62	4.4040	
		Sum	17.49	17.49	14.67	14.70			15.8169	40.4003

Difference 24.5834.

The column marked D. M. D. contains the sum of each meridian distance added to the preceding; or the double meridian distances. We have, therefore, in this column the sum of the parallel sides of the trapezoids, and the bases of the triangles to be computed. Multiplying these by the corresponding northings and southings, we obtain the area, written on the same line, in the area columns. By the process the double area is in each case computed; taking half of the result, we have for the area required,

12 A. 1 R. 7 P.

140. Since in the course of the operations we return to the same point of the field from which we started, the sum of the northings, it is evident, should be equal to the sum of the southings, and the sum of the eastings to the sum of the westings. When this is not the case, some error must have occurred either in the operations in the field, or in the computations. And the work must be re-examined, beginning with the most probable source of error, until it is detected.

If the error, however, is trivial, it may be regarded as the result of unavoidable imperfections in the survey and in the calculations.

In this case it should be averaged upon all the sides by the following proportion,

As the perimeter of the field,
Is to the length of one of the sides,
So is the error in latitude or departure,
To the correction corresponding to that side.

This correction must be added when applied to a column in which the sum of the numbers is too small, and subtracted in the case where the sum of the numbers is too great.

In the present example, there is a difference of 3 links between the sum of the eastings and that of the westings. By the rule just stated we find the corrections for the sides and the corrected eastings and westings as follows.

	Correction.	Cor. E.	Cor. W.
1.	.003	1.673	
2.	.005	5.005	
3.	.004	6.954	
4.	.005	1.056	
5.	.005		3.045
6.	.007		11.643
		14.688	14.688

Additional columns may be ruled in the table for the corrected latitudes and departures. The corrections should be made and the columns balanced, before proceeding to the computation of the areas.

141. In the example we have considered, the north areas all lie without the field, and do not, therefore, comprehend any part of it. It will sometimes happen, however, as in the case of a *re-entering* angle, or one which returns into the field, that a north area will include a part of the field. But in this case the portion thus included will enter twice into the south areas; and the rule for finding the area of the field will still be true.

142. In the calculations above we have found the latitudes and departures by aid of the logarithmic sines, cosines &c. In general, especially if the number of figures in the distances does not exceed three or four, it will be best to use the natural sines.

Radius being unity, the natural cosines and sines are decimals, and may be regarded as the latitudes and departures for the distance unity. To find, then, the latitude and departure for a given course and distance, we multiply for the latitude the cosine of the course by the distance; and for the departure the sine of the course by the distance.

To find, for example, the latitude and departure when the course is 32°, and the distance 9, we have

$$\begin{array}{rr} \text{N. cos } 32^\circ = .84805 & \qquad \text{N. sin } 32^\circ = .52992 \\ 9 & 9 \\ \hline 7.63245 & 4.76928 \end{array}$$

which gives 7.63 for the latitude, and 4.77 for the departure; true to the second decimal place or nearest link, which is sufficient. In general, to attain this degree of accuracy, it will be sufficient to take out the natural sines to the third or fourth place, observing to increase the right hand figure of the part employed by 1, when the left hand figure of the part omitted exceeds 5. Thus, in the example above, we may take .53 for the natural sine, which being multiplied by 9 gives 4.77 as before.

To facilitate the calculations of latitude and departure, tables have been prepared containing the latitude and departure for every distance from 0 to 100, and for every course in degrees and quarters of degrees from 0 to 90°. Tables of this description are called *Traverse tables.* They are usually calculated to two places of decimals only; and as there are various sources of error in the use of them, it is better that the surveyor should make his own computations entire.

143. The method to be pursued will require no further explana-

tion. The following examples will serve as an exercise for the learner.

Ex. 1. To find the contents of a field from the following field notes.

1.	N. 20° E.	8.50 ch.
2.	S. 76° E.	10.00.
3.	S. 25° W.	13.50.
4.	N. 67° W.	6.00.
5.	N. 18° W.	4.55.

Ans. 11 A. 1 R. 20 P.

Ex. 2. To find the contents of a field from the following field notes.

1.	N. 15° E.	8. ch.
2.	N. 55° E.	6.
3.	S. 20° E.	7.
4.	S. 50° W.	4.
5.	S. 25° E.	8.
6.	N. 61½° W.	11.

Ans. 8 A. 1 R. 8 P.

Rectangular method of Plotting.

144. The position of a point is determined, most conveniently, by referring it to two lines which make a right angle with each other; and are given in position.

Thus, the position of the point B, in the figure below, is determined by the perpendiculars B*b*, B*p* let fall from this point upon the two indefinite lines AN, AE which make a right angle NAE with each other. The right lines AN, AE are called *axes*; the perpendiculars B*b*, B*p*, or their equals A*p*, A*b*, are called the co-ordinates, as we have seen, of the point; and the point A, in which the axes intersect, the *origin* of the co-ordinates.

145. Let it now be proposed to plot the field, art. 139.

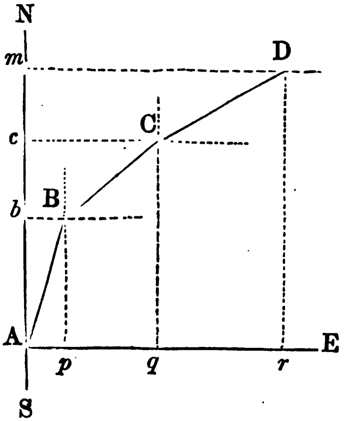
Let the corner A of the field be assumed as the origin of the co-ordinates; and let the first meridian NS, and the east and west line AE, drawn through A, be taken as the axes.

From the calculations entered in the table for computing the area, it will be easy to determine the co-ordinates of the corners B, C, &c. of the field. Thus the co-ordinates of the corner B will be, it is evident, its latitude A*b*, and the meridian distance *b*B; the co-ordinates of the point C will be the latitude A*c* of C, reckoned from A, and the meridian distance *c*C; and so on.

Referring to the figure, art. 139, the latitude A*c* of C, reckoned from A, is equal, it is evident, to the sum of the latitudes A*b*, B*d*; the latitude A*m* of D to the sum of the latitudes A*c*, C*n*; that of E, equal to A*r*, to the difference of the latitudes A*m*, *o*E; and so on.

From the calculations, therefore, we shall have the following for the co-ordinates of the corners B, C, D, &c. of the field.

Corners.	Lat.	M. D.
B	5.14	1.67
C	10.89	6.67
D	13.70	13.62
E	5.75	10.57
F	–3.79	11.62



To plot the field we draw the indefinite line NS representing the first meridian; and assuming upon this the point A as the first corner of the field, we draw next the indefinite east and west line AE. The lines NS, and AE will be the axes. To find the corner B, we set off on AN a distance $Ab = 5.14$, and on AE a distance $Ap = 1.67$. Through the points *p* and *b*, we then draw lines parallel to AN, AE respectively. The point of intersection B of these two parallels, will be the corner B of the field. We proceed in the same manner to find the remaining corners of the field. Joining next the points thus determined, we obtain a plot of the field.

146. The superiority of the method will readily be seen. To find the point C, for example, in the ordinary way, we lay off first the line AB, according to the prescribed course and distance; and next the line BC, the extremity C of which is the point sought. These operations are liable to considerable error from the imperfection of instruments. Any error, moreover, in laying down the line AB will be propagated to the line BC, and so on. Whereas, the co-ordinates A*c*, *c*C may be computed with as much exactness as we please, and the parallel lines may be drawn with great accuracy. Furthermore, any error in the plotting of the point B will not necessarily affect the point C.

We have assumed, thus far, that the lines NS are true meridians determined by the compass. So far as the computation of the area or the plotting of the field is concerned, this is not necessary. A line may be taken, as the first meridian or base of the operations, making any angle at pleasure with the side AB of the field, provided the angles of the field are known.

Ex. To plot a field and find its contents from the following field notes.

1. N. 19° E. 8.40 ch
2. S. 76° E. 5.90.
3. S. 17° E. 8.85.
4. N. 61° W. 4.60.
5. S. 24° 30′ W. 6.15.
6. N. 41° 20′ W. 6.88.

Ans. 7 A. 3 R. 29 P.

SECTION IV.

LAYING OUT, AND DIVIDING LANDS.

147. We have seen the operations required in the field, the method of forming a plot of a survey, and of computing its area. We proceed next to consider the manner of laying out and dividing lands according to prescribed conditions.

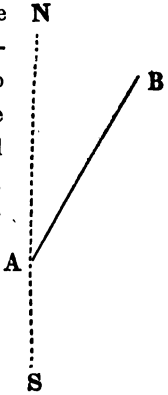
Laying out lands.

Ex. A certain tract of land, situated in Brunswick, Me. is conveyed by deed in the following terms:

"Beginning at a stake and stones on the Easterly line of the Harpswell road, and at the Northwest corner of land of John Doe; and running thence N. 23° E. 22 rods on said road; thence S. 67° E. 50 rods; thence S. 23° W. 32 rods; thence N. 62° W. by land of said Doe, to the first mentioned bound."

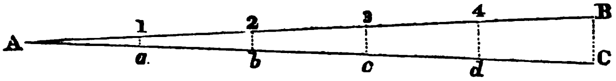
It is required to mark upon the ground the bounding lines of this field.

Having ascertained the point A of beginning, the surveyor places his compass at this point; and directs the sights, so that the needle, when it comes to a state of rest, will point to 23° on the left of the north point on the compass box. The sights will then be in the direction of the line first to be run. Marking the direction of this line by a stake or stakes, a distance of 22 rods is then measured upon it from A to B. The corner B and the first side AB of the field is thus determined. Proceeding in like manner, the remaining corners and sides of the field may be determined; and the entire boundaries of the field fixed by fences or other permanent marks.



148. In running lines, various obstacles occur which it is required to overcome.

Let it be required to run a line between two points A, and B, neither of which, by reason of a thick wood, can be seen from the other, and the course between which is not known.



In this case the surveyor runs a line AC, as near as he can judge to the true line, noting its course, and putting down stakes upon it at regular intervals A*a*, *ab*, &c. at the end of every chain for example. Having arrived opposite to B, he next measures from B a perpendicular BC upon the line AC. Then calculating in the triangle ACB the perpendiculars *a*1, *b*2, *c*3, &c, he puts down stakes at the points 1, 2, 3, &c. These will be upon the line AB which is thus determined.

The line AC from which AB is determined, is called a *random line*.

To find next the course between A and B, we have in the triangle ACB,

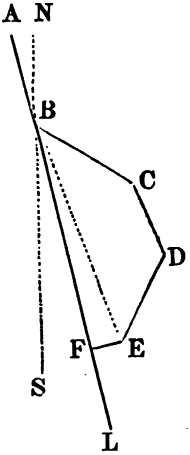
$$AC : BC :: R : \tan BAC.$$

The angle BAC being obtained by this proportion, the required course will then be easily found.

149. If the obstacle obstructing the sight upon the line is a house, for example, the following process may be employed.

Having marked the line to a point near the obstacle, an offset of sufficient length is made at right angles with the line. Proceeding from this on the same course with the line, until the obstacle is passed, an offset at right angles is again made, equal to and in the opposite direction from the former. The surveyor is then upon the line to be marked, and may proceed as before.

150. If the obstacle is of greater extent, as a swamp or pond, the following course may be pursued.



Let the obstacle be a pond across which the line AL is to be run. On arriving at the edge B of the pond, the surveyor runs, from this point, as many courses BC, CD, &c. round it, as may be convenient or necessary; until at E, for example, he judges that he is nearly upon the line AL. By means of the difference of latitude and departure of the several courses BC, CD, &c. he then calculates, as in art. 136, the latitude and meridian distance of the point E from B. With these he next calculates the bearing SBE, and the length of the line BE. Supposing a perpendicular EF let fall from E upon AL, this perpendicular, and its bearing, may then be found from the triangle BEF. Finally running from E a line EF of the length and bearing thus determined, the point F will be upon the line AL which it is required to run, and from which the work may now be continued.

The line BF, it is evident, may be calculated from the triangle BEF, and thus the whole distance AL be laid off.

Ex. A surveyor in running a line, the bearing of which is S. 49° E. comes to a swamp across which the line is to be carried. He then makes his way around the head of the swamp by the following courses and distances;

No.	Bearing.	Distance.
4	S. 24° 15′ W.	4.39
3	S. 20° 15′ E.	3.87
2	S. 74° E.	7.25
1	N. 57° E.	4.04

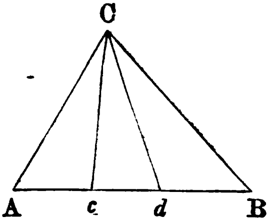
it is required to find the bearing and distance, from the point at which he has arrived, of the perpendicular upon his line.

Ans. Bearing S. 37° W. Dist. .80 ch.

DIVIDING LANDS.

151. In addition to laying off the boundaries of lands and estates, the surveyor may be required to divide a certain tract or parcel of land among different proprietors or heirs; and determine upon the ground the particular portion of each.

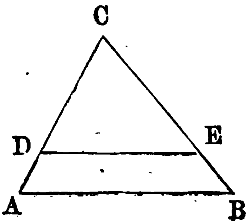
In order to this, it will be best, in general, first to make a plot of the portion of land to be divided. The surveyor will next determine, by aid of geometry or by calculation, what operations in the field are necessary to effect the required division; after which the operations themselves will be easily performed. We give a few examples.



Ex. 1. It is required to divide a triangular field ABC equally among three persons, by lines proceeding from the vertex C of the triangle.

Since triangles of the same altitude are to each other as their bases, it will be sufficient, it is evident, to divide the base AB into three equal parts, and then to draw the lines Cc, Cd to the points of division c and d.

If the parts, instead of being equal, are in any proportion to each other, as that of m, n, and p for example, it will be sufficient to divide the base into parts in the same proportion; and then to draw lines from the vertex C to the points of division, as before.



Ex. 2. Let it next be required to divide a triangular field ACB into two parts, in the ratio of m to n, by a line parallel to the base.

Let DE be the dividing line; then the part CDE being similar to the whole, we shall have

$$m+n : m :: \mathrm{CA}^2 : \mathrm{CD}^2;$$

whence $$\mathrm{CD} = \mathrm{CA}\left(\frac{m}{m+n}\right)^{\frac{1}{2}}.$$

Having found CD by this formula, the division may then be easily effected. Thus, let it be required to divide the triangle into two equal parts, and let the side CA be 15 ch.; then

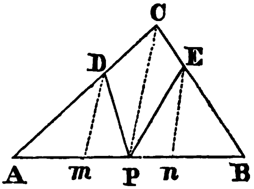
$$CD = CA\sqrt{\tfrac{1}{2}} = 15 \times 0.707 = 10.605 \text{ ch.}$$

Laying off next 10.605 ch. from C to D, and running from D a line parallel to AD, we shall have the division required.

If it be required to divide the triangle into three equal parts, by lines parallel to AB, putting D′ for the second point of division upon CA, we shall have

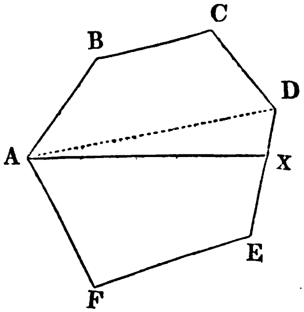
$$CD = CA\sqrt{\tfrac{1}{3}}, \text{ and } CD' = CA\sqrt{\tfrac{2}{3}}.$$

Ex. 3. Let it now be required to divide a triangular field ABC into three equal parts, by lines proceeding from a point P in the base.



From geometry, we derive the following process for the solution of this problem. Divide the base AB into three equal parts; and let m and n be the points of division. Join PC, and from m and n draw lines mD, nE, parallel to PC. Draw next the lines PD, PE; and the triangle will be divided by these lines as required.

Ex. 4. In the irregular field ABCDEF let it be required to draw a line from the corner A, so as to divide the field into two equal parts.



From inspection of the plot it appears that the dividing line AX will fall upon the side DE. Draw the line AD from A to D, and calculate the part ABCD, which suppose less than one half of the field. Subtracting this area from one half the area of the field, we shall have the area of the triangle ADX, which must be added to ABCD to effect the division required. To find this triangle we have the side AD, the angle ADX, and the area A. Putting x for DX the base of the required triangle we have

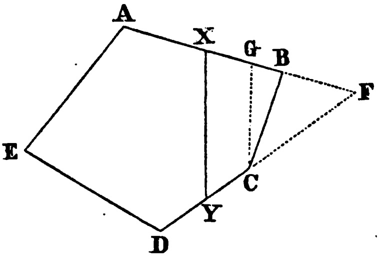
$$\tfrac{1}{2}x \times \mathrm{AD} \times \sin \mathrm{ADX} = \mathrm{A};$$

whence $$x = \frac{2\,\mathrm{A}}{\mathrm{AD} \sin \mathrm{ADX}} = \mathrm{DX}.$$

Laying off from D the line DX of the length thus determined, it will remain only to run the line AX, and the division is effected.

Ex. 5. The direction of a line being given, it is required to find its position, so that it may cut off a given area from a given field.

Let ABCDE be the given field. It is required to run a line XY from some point X in the side AB, and in a given direction, so that the part BXYC cut off, shall be of a given magnitude.



Produce the sides AB, DC until they meet in F; then from the bearings of the sides the following angles will be known, viz. FBC, BCF, and FCB.

Draw CG, in the given direction, parallel to XY; the angles CBG, BCG, CGB will then be known.

The triangle FBC gives the proportions

sin BFC : BC : : sin FBC : CF,

and sin BFC : BC : : sin FCB : FB;

from which the sides FC, FB may be found; and next the area of the triangle FBC.

Again in the triangle GBC, we determine the sides GB, GC; and also the area of the triangle. We thus know the area of the triangle FGC; and it remains only to find the line GX, so that if a line XY be drawn through X parallel to GC it will give an area GXYC equal to the difference between the area BXYC required to be cut off and the known area GBC. The question is thus reduced to problem 2d; and may, therefore, be readily solved.

The preceding problems illustrate the general course to be pursued in the division of land. The solution of the cases which occur in practice must depend upon the skill of the surveyor, and his general acquaintance with the principles of geometry and algebra.

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SECTION V.

VARIATION OF THE NEEDLE. PUBLIC LANDS.

The compass is a convenient instrument for running courses in woods; but it is subject to many imperfections, and its use in determining boundaries has been a prolific source of uncertainty in respect to them.

VARIATION OF THE NEEDLE.

152. The principal cause of imperfection in the compass arises from the variation, or deviation of the direction in which the needle settles from a true north and south line. This is different in different places, and varies, moreover, in the same place from year to year. The variation of the needle at London may be taken as an example of the changes of variation in the same place. Having for some years previous been increasing easterly, the variation at London reached in 1580, 11° 17′ east. From this it began to decrease, until in 1657 it became 0; and the needle pointed to the true north. From this the variation commenced in a westerly direction; and increased in that direction until, in 1815, it reached 24° 27′ 18″ west, when it began to decrease; and the direction of the needle is again approaching more nearly a true north and south line.

The following table will show the variation in 1840 for several places in the United States.

Burlington, Vt.	9° 27′ W.	Buffalo, N. Y.	1° 37′ W.
Boston, Mass.	9° 12′ W.	Cleaveland, O.	0° 19′ E.
Albany, N. Y.	6° 58′ W.	Detroit, Mich.	1° 56′ E.
New Haven, Ct.	6° 13′ W.	Charleston S. C.	2° 44′ E.
New York City	5° 34′ W.	Cincinnati, O.	4° 46′ E.
Philadelphia,	4° 8′ W.	Mobile, Ala,	7° 5′ E.
Washington City,	2° 0′ W.	St. Louis, Mo.	8° 37′ E.

In the northern states the variation has increased since 1840, about 5 minutes annually; in the middle states from 3 to 4 minutes. In the western states it has decreased from 3 to 4 minutes, and in the southern about 2 minutes annually.

There is also a small daily variation depending upon the temperature, which is greatest in the easterly direction about 7 o'clock in the morning, and in the westerly direction about 2 o'clock in the afternoon. During the night the needle is stationary. There is, moreover, a small annual variation depending upon the same cause. Neither of these are, however, of sufficient amount to be of any consequence in the common operations of surveying.

MERIDIAN LINES.

153. In order to determine the variation at any place, it is necessary to trace a true meridian with which the direction of the needle may be compared. This is properly a problem of astronomy, for which the learner is not supposed at present to be prepared. The following, however, in the main may be easily understood, and is the best practical method in use.

The method is by observations upon the pole star (*Polaris*).

If the pole star were precisely at the pole, or the point where the axis of the earth produced pierces the heavens, a line drawn from any point on the surface of the earth in the direction of the star, would be a true meridian. But the star is at a mean distance of about 1° 30′ from the pole, about which it revolves in 23 hours 56 minutes.

There are two ways in which the direction of the pole may be determined by observations upon the star. 1°. By observing the star at the moment when it is on the meridian either above or below the pole. 2°. By observing it when at its greatest distance east or west of the pole, called its greatest eastern or western *elongation.*

When the star is near the meridian, its apparent motion in an easterly or westerly direction is rapid, and the precise moment when it is on the meridian is, therefore, with difficulty observed. But when the star is near its greatest eastern or western elongation, its apparent motion, east or west, is slow; and when at either of these points it does not perceptibly change its direction for some fifteen or twenty minutes. This furnishes opportunity for careful observation. And the time of its greatest eastern or western elongation having been carefully noted, the star in 5 hours and 59′ after will be on the meridian above or below the pole as the case may be. A line drawn at that time in the direction of the star will be a true meridian.

But these methods both requiring the means of accurately determining the time, which are not usually at command, the following is to be preferred; viz, 1°, observe the star, as before, at its greatest eastern or western elongation, and mark its direction; 2°, calculate the angle which the true meridian makes with this direction. This will furnish the means by which it may be laid off.

For the calculation required we have a right angled spherical triangle, in which the complement of the latitude of the observer is the hypothenuse, the angle at the star is a right angle, and the star's polar distance the side opposite the angle required. To obtain this angle we have the following proportion,

Cos Lat : R : : sin Star's polar dist. : sin of required angle.

The bearing of the star, or its angular distance east or west of the pole, as thus determined, is called its *azimuth.*

154. The elongations of the star which answer the purpose, are those which occur in the night. The times in which they occur, approximately at least, it is convenient to know beforehand. They may be found by means of the following tables, which will be sufficient for this purpose for twenty years to come.

EASTERN ELONGATIONS.

Days.	April.	May.	June.	July.	Aug.	Sept.
	H M.	H M.	H. M	H. M	H. M.	H. M.
1	18 18	16 26	14 24	12 20	10 16	8 20
7	17 56	16 03	14 00	11 55	9 53	7 58
13	17 34	15 40	13 35	11 31	9 30	7 36
19	17 12	15 17	13 10	11 07	9 08	7 15
25	16 49	14 53	12 45	10 43	8 45	6 53

WESTERN ELONGATIONS.

Days.	Oct.	Nov.	Dec.	Jan.	Feb.	March.
	H M	H. M.	H. M.	H. M.	H. M.	H. M.
1	18 18	16 22	14 19	12 02	9 50	8 01
7	17 56	15 59	13 53	11 36	9 26	7 38
13	17 34	15 35	13 27	11 10	9 02	7 16
19	17 12	15 10	13 00	10 44	8 39	6 54
25	16 49	14 45	12 34	10 18	8 16	6 33

The time is *astronomical,* and is reckoned from noon. Thus 18 H. 18 M. is 6 H. 18 M. after midnight; or 18 minutes past six in the morning.

Tables also may be computed giving the azimuth for different latitudes for a succession of years. But it is better for the surveyor, by the proportion above, to calculate the azimuth corresponding to his latitude and time of observation. In order to this, it will be necessary to know the distance of the star from the pole at the time of the observation. This distance is variable and is now decreasing. We give below the polar distance of the star, Jan. 1st, for a few years, together with the annual variation.

Year.	Polar distance.	An. Var.
1852	1° 28′ 46″.18	19″.256
1853	1° 28′ 26″.93	19″.248
1854	1° 28′ 7″.68	19″.241
1855	1° 27′ 48″.44	19″.234

The annual variation may be reckoned at 19″, sufficiently near for practical purposes for years to come.

Ex. The latitude of Brunswick (Bowd. Coll.) Me, is 43° 53′; and the eastern elongation of the star was observed Sept. 1st, 1854. What is the azimuth of the star?

Cos. Lat. 43° 53′	9.857786	
Radius	10.000000	
Sin. Pol. dist. 1° 27′ 36″	8.406183	
Sin. 2° 1′ 33″	8.548397	Ans.

155. It remains only to explain the method of observing the star. This may be done with the theodolite as follows.

At fifteen or twenty minutes before the time of the elongation to be observed, as found in the table above, the instrument should be placed at the point from which the line is to be drawn. It should then be properly levelled and sufficient light directed upon the cross threads of the telescope to render them visible.

This being done, the observer will next turn the telescope until the star appears directly behind the vertical thread, and will continue to follow the star until it has reached the point of greatest elongation. At this point it will appear stationary for a time, and then gradually leave the thread of the telescope in a direction *opposite* to that in which it has been previously moving. The axis of the telescope will thus be left in the direction of the star at the observed elongation.

To mark this direction, an assistant will next hold a lamp, at some convenient distance, in front of the telescope, moving it to the right or left as may be required, until it appears to the observer directly behind the vertical thread of the telescope. The point thus intersected will then be carefully marked, and a line drawn from the centre of the instrument to this point will be in the direction required.

To mark the meridian line we now calculate the azimuth of the star for the given time and place. Marking a line, which shall make with the line already determined an angle, to the right or left as the case may require, equal to the azimuth found, we shall have the meridian required.

To throw light upon the cross threads of the telescope, a board about a foot square may be prepared movable up and down upon a staff, with a small shelf at the lower edge on which to place a lamp. The board should be covered with white paper, and a hole perforated at the centre sufficiently large for the star to be seen through the telescope. The board being properly arranged before the telescope, the threads will be sufficiently illuminated, while the star at the same time is seen.

Instead of laying off the azimuth angle by the theodolite, after marking the bearing of the star we may assume any given distance upon this bearing as a base, and calculate with this distance and the azimuth, the perpendicular distance to the true meridian; by means of which the meridian may be determined as before.

156. In case a theodolite is not at hand, the following method of observing the star may be employed.

Let two stakes be driven in the ground about four feet apart in a direction east and west; and on the top of these let a joist be nailed, smooth on the upper surface, which should be made perfectly level. At about twelve feet in front, let a plumb-line be suspended from the top of an inclined stake, of sufficient height for the star to be seen behind the line from the horizontal bar. The plumb should be swung in a vessel of water to keep it from vibrating.

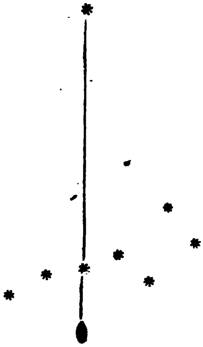
Upon the horizontal bar let a sight vane of the compass be placed, and moved to the right or left, as may be required, until the observer, looking through the slit in the vane, sees the star, at its greatest elongation, directly behind the plumb-line. This being done, the direction of the star at its greatest elongation will be known as before.

In this operation it will be necessary to have the plumb-line lighted; which may be done by an assistant holding a lamp near it.

157. An approximate meridian line may be easily drawn in the following manner.

The pole star is very nearly on the meridian when it is in the same vertical plane with the star Alioth in the tail of the great bear, or that which lies nearest the quadrilateral.

The same arrangements being made for observing, as in the preceding article, the pole star and Alioth are then followed until both appear, at the same time, directly behind the plumb-line. The direction thus obtained will be very nearly a true meridian.



158. Having established a meridian line, the variation of the needle is easily observed. All we have to do is to place the compass upon the line, and note the angle which the needle makes with it.

A meridian line should be established in some convenient place in every town, by means of which the variation of the needle may, at any time, be ascertained. And in all conveyances of real estate, the boundaries of which are determined by the compass, the variation of the needle at the time the boundaries were run, should be carefully noted.

If then, at any future period, a line is required to be run anew, the surveyor will have the means by which to determine its true direction.

SURVEY OF THE PUBLIC LANDS.

159. To divide the public lands into portions convenient for sale, and to fix with precision the position and bounding lines of such portions, the General Government adopted in 1802, the following method of survey, proposed by Coloned J. Mansfield then surveyor-general of the North-Western Territory.

1°. Through or near the middle of the tract surveyed a meridian line is run, and designated by permanent marks. This is called the *principal meridian.* Parallel to and on both sides of this, other meridians are run, at the distance of every six miles.

2°. At or near the middle of the principal meridian and perpendicular to it, an east and west line is run, called the *principal parallel* or *base line.* On both sides of this, at the distance of every six miles, other parallels are also run; dividing the territory into equal squares, six miles on a side, and containing 36 square miles each. These squares are called *townships.*

3°. Each township is again divided into smaller squares called *sections*, by lines run parallel to the meridians and parallels, at the distance of one mile from each other. A section, therefore, contains one square mile or 640 acres. The sections are again divided into quarters, eighths or sixteenths.

The townships which lie along the same meridian, are called *ranges.*

The ranges are numbered east or west of the principal meridian. The townships are numbered north or south of the base line. The sections are numbered, beginning at the northeast corner of the township, from east to west and west to east progressively, to 36 the last number, which is found in the south east corner of the township.

A particular section is denoted thus; section 10, township 3 north, in range 5 east.

To find the section, therefore, we seek the fifth range of townships east of the principal meridian; and the third township in this range north of the base line will be the township. The tenth section in this township will be the section sought.

New principal meridians are run and properly numbered when necessary. New principal parallels are in like manner also run.

160. In consequence of the convergency of the meridians, the townships, laid down as above, will not be perfect squares, the north bounding lines being less than the corresponding south ones. To prevent the excess or deficiency in the area of the townships arising from this source from accumulating, new parallels are measured at suitable distances on each side of the base line, called *standard parallels*, or correction lines.

The standard parallels north of the base line serve as bases for the townships north of them; and those south of the base line for the townships south of them.

In the original surveys the townships are indicated by posts set in the ground, square at the top, and properly marked and numbered. When a post is designed to indicate the corner of four townships, diagonal lines are drawn across the top, and the post is set so that these lines will range, one due north and south, and the other due east and west. When a post marks the corners of but two townships, as will be the case on the standard parallels, it is set so that its faces will range due north and south, and due east and west.

The corners of the sections and quarters of sections, are also indicated by posts or other permanent marks.

In running out the sections in a township, the excess or deficiency of area in the township, arising from the want of parallelism in the meridians, is usually thrown upon the most westerly tier of sections; and the precise area of each of these is determined and suitably noted. The other sections will then contain each the required number, or 640 acres.

The principal meridians are numbered from east to west, and, with the base lines, are characterized by peculiar marks. Thus, the surveys in Wisconsin are all referred to the fourth principal meridian, which starts from the mouth of the Illinois river. The base line is the southern boundary of the State, so far as it borders on the State of Illinois.

The standard parallels are run at a greater or less distance from each other according to the convergency of the meridians. Thus in the surveys in Oregon, it is proposed to run the standard parallels north of the Columbia river at a distance of four townships or 24 miles from each other; and south of that river at the distance of five townships or 30 miles from each other. In general the distance between the parallels should not exceed sixty miles.

The principal meridians and parallels are determined by astronomical observations. The subdivisions are run by the compass.

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SECTION VI.

LEVELING.

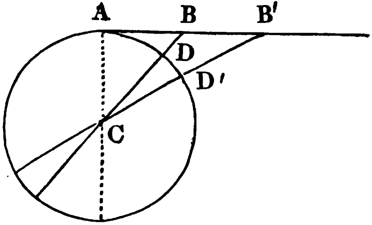
161. The operations to which we have thus far attended relate chiefly to the determination of the bounding lines and other similar features of fields or estates. There are other objects, however, for which surveys are required, such as roads, railroads, canals, and other public works. For these objects, it becomes necessary to determine the differences of level between different points of the earth's surface. The mode of doing this we proceed next to explain.

162. Two points are said to be on a level when they are equally distant from the centre of the earth.

The surface of a fluid at rest is a level surface; since all its points are at equal distances from the centre of the earth. Any two points, therefore, are on a level, when they are equally distant from the surface of a tranquil fluid; supposed to be situated immediately above or below them.

A level surface, moreover, is one that is everywhere perpendicular to a plumb-line, or the radius of the earth considered as a sphere. This is called a *true level.*

163. A plane surface perpendicular to the radius of the earth at one point only, is called an *apparent level.*



Thus if a line AB is drawn perpendicular to the radius AC of the earth at A, any two points A and B of this line are on the same apparent level. And if a line BC be drawn to the centre of the earth intersecting its surface at D, BD will be the difference between the true and apparent level of the point D, referred to a tangent at A.

To find this difference we have (Geom.)

$$AB^2 = BD \times (BD + 2\,DC).$$

But for any distance we have occasion to measure, BD is very small compared with 2 DC, or the diameter of the earth. BD may, therefore, by the side of 2 DC, be neglected without sensible error. The arc AD and its tangent AB, moreover, will not, for small distances, essentially differ; we have, therefore

$$AD^2 = BD \times 2\,DC; \qquad (1)$$

whence

$$BD = \frac{AD^2}{2\,DC}.$$

That is, the difference between the apparent and true level for any distance, is equal *to the square of the distance divided by the diameter of the earth.*

Ex. Let AD = a statute mile, or 5280 feet; and 2 CD, the diameter of the earth, = 7912 miles; we have

$$BD = \frac{(5280)^2}{7912 \times 5280};$$

or, performing the operations, we have

$$BD = 8.0076 \text{ inches.}$$

Thus, the difference between the apparent and true level for a distance of one mile is 8 inches.

If we take any other distance AD′, we have

$$AD'^2 = B'D' \times 2\,DC; \qquad (2)$$

comparing the equations (1) and (2), we obtain

$$BD : B'D' :: AD^2 : AD'^2.$$

Whence, *the difference between the apparent and true level for different distances, is as the square of the distance.*

Ex. What is the difference between the apparent and true level for two points distant 2½ miles from each other.

By the proportion above, we have

$$1^2 : 2\tfrac{1}{2}^2 :: 8 \text{ inch.} : 50 \text{ inch.}$$

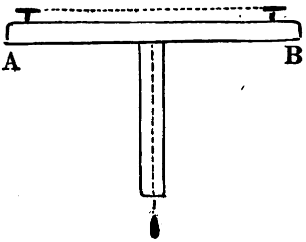
To find the difference of level in inches, therefore, it will be sufficient to multiply the square of the distance in miles by 8; or, which returns to the same thing, *two thirds the square of the distance in miles will give the difference of level in feet.*

LEVELING INSTRUMENTS.

We pass next to explain the instruments commonly employed for determining a level.

Plumb-line Level.

164. A very simple instrument for determining a level may be easily constructed, as follows.



Let AB be the straight edge of a board, and CD a bar attached to the middle of it; so that a right line drawn through the middle of the bar, will be exactly at right angles with the edge AB. If a plumb-line be now suspended at the point of meeting, the edge AB of the board will be level, when the plumb-line exactly covers the line drawn upon the bar. If then two sights are placed near the extremities A and B of AB, at equal distances above it, the line of sight across these will be a level line.

Water Level.

165. Another instrument may be easily constructed for determining a level, derived from the principle that water in a bent tube will always rise to the same height at each end, however the position of the tube may be varied.

Let a tube bent up at each end be partially filled with water; a line coincident with or parallel to the surface of the water at each end of the tube, will be a level line.

Spirit Level.

166. The essential part of a spirit level is a glass tube slightly curved and nearly filled with alchohol, just leaving an air-bubble within, which will always seek the highest part of the tube. This is imbedded lengthwise and with its convexity upward, in the upper surface of a bar eighteen inches or two feet in length. The under surface of the bar may then be so adjusted, that the air-bubble will always remain at the centre of the tube, in whatever position the bar is placed in a horizontal plane. If sights are placed at each end of the bar, and at equal distances above its under surface, the line of sight across these will be level, when the bubble is at the centre.

The Engineer's Spirit Level.

This is the only convenient and accurate instrument for determining a level. Its essential parts are; 1°, a telescope with cross threads in the axis like the theodolite, and an attached spirit level, so adjusted that when the air bubble is at the centre of the level, the axis of the telescope is in a horizontal line. 2°. A tripod with two brass plates, to the lower one of which the legs are attached. The upper plate is connected with the lower by means of a ball and socket joint; and is capable of being moved upon it in all directions by four screws, called the leveling screws. The telescope, when in use, is placed upon the upper plate, having a free motion about a pin at the centre. By means of the leveling screws, the bubble may be brought to remain at the centre of the attached level; and thus the axis of the telescope continue in the same horizontal plane, in whatever direction the telescope is turned.

Leveling Staves and Chain.

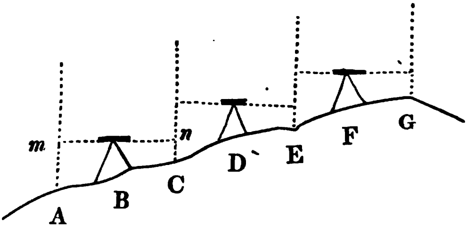
167. The staves employed are usually ten feet in length, divided into feet, and tenths and hundredths of a foot. A target is made to slide up and down the staff, with a line across it at right angles to the staff, marked by different colors on opposite sides for the sake of distinctness.

The chain commonly employed is 100 feet in length, divided into 100 links. A link is, therefore, one foot.

DIFFERENCE OF LEVEL.

168. Let it now be proposed to find the difference of level between any two points A and C.

Place the instrument at some convenient point B between A and C, and bring it to a level. Let an assistant hold a staff in a vertical position at A; then turn the telescope toward it, directing the target to be slipped up or down, as may be required, until the horizontal thread of the telescope intersects the cross line of the target at *m*, for example. A*m* will then be the distance of the line of intersection from A, or the distance of A below the horizontal line *mn*. Let the assistant now remove the staff to C; and the telescope being turned toward it, find in like manner the distance C*n* of C below the horizontal line *mn*. The difference between A*m* and C*n* will be the difference of



level between the points A and C. Thus, suppose $Am = 10.75$ feet, $Cn = 7.30$; the difference of level between A and C is 3.45 feet.

If the points are at a considerable distance from each other as A and G, for example, several intermediate stations as C, E, &c. may be necessary, which are taken at convenience. Having found in this case the difference of level between the points A and C as before, the instrument is moved forward to some convenient position D between C and E; and the difference of level between these points is determined in like manner, and so on to G.

169. In this series of observations the sights in the direction from the instrument toward A, the point of departure, are for the sake of distinction called *back sights*, and those toward G are called *fore sights*. If the ground is ascending, the back sight will be greater than the fore sight, if descending it will be less. In either case the difference between the fore and back sights will be the difference of level between the two consecutive sights, and should be marked + or —, according as the ground is ascending or descending.

The field book may be kept as follows.

No.	B. S.	F. S.	Diff.	Total.
1	4.52	1.32	3.20	3.20
2	5.48	2.53	2.95	6.15
3	6.72	4.25	2.47	8.62
4	2.43	7.53	— 5.10	3.52
5	1.20	6.45	— 5.25	—1.73

In the first column is the number of the station, in the second the back sights, in the third the fore sights. In the next the differences, marked + or —, as the ground is ascending or descending. In the last column is the total or sum of the differences, which shows the difference of level between each successive station and the first, and finally the difference between the points required.

In the present example the last point is 1.73 feet below the starting point.

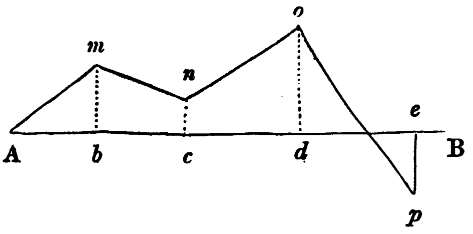
In the operations the level should be placed as nearly as possible at the centre between the stations; since in this case no correction will be necessary for the difference between the apparent and true level.

Profile.

170. The column marked total in the preceding table contains, it is evident, the height or vertical distance of each station from a horizontal line passing through the point at which the work is commenced. If then the distances between the stations are also measured, a profile, or side view of the ascents and descents of the ground between the extreme stations, may be drawn.

In measuring the distances between the stations, the chain should be kept horizontal. We thus obtain, without the necessity of calculation, the projections of the distances upon a horizontal plane; which, as it will be easy to see, are the distances required in plotting the profile.

Let the distances between the stations be 25, 30, 50, and 40 feet respectively; and the total differences of level corresponding, be 10, 7, 12 and — 3 feet. It is required to draw the profile.



Upon an indefinite line AB representing a horizontal line through the first station A, set off A*b*, *bc*, *cd*, *de* equal respectively to the distances 25, 30, 50 and 40; then at the point *b* erect the perpendicular $bm = 10$, the first difference of level; and from *c*, the perpendicular $cn = 7$, the second difference; and so on. The last perpendicular, being negative, is set off below AB. Next through A and the extremities *m*, *n*, *o*, *p*, of the perpendiculars, draw the broken line A*mnop*, which will be the profile required.

171. The differences of level are usually very small in comparison with the distances between the stations. If the distances and heights were, therefore, plotted to the same scale, the latter would be scarcely visible. On this account it is usual to plot the heights on a much larger scale than the distances. The proportion usually employed is 1 to 20. Thus 5 feet in the differences of the levels will correspond to 100 feet between the stations. The profile in this way is greatly exaggerated; but it occasions no error, since it is the differences of level only that we wish to know at different points.

Ex. A gentleman wishing to determine to what height water can be introduced into his house by a pipe from a distant spring, took the following levels between them.

No.	Dist.	B. S.	F. S.
0	1.00	1.25	.20
1	1.30	.10	2.80
2	1.50	3.50	.35
3	1.20	1.15	3.10
4	.90	1.00	2.80
5	1.40	1.20	.10
6	1.25	.30	2.20
7	1.00	3.90	.50
8	1.15	.20	3.40
9	1.40	.45	4.80
10	2.50	.60	2.90
11	2.00	1.50	.50

Performing the calculations, it will be found that the water will rise to a height of 8.50 ft. from the ground at the house. Let the learner draw the profile, and show at what points and to what depth the pipe must be sunk in the ground, in order that no part of it may be above the level of the water in the spring.

172. A permanent mark, called a *bench mark*, is usually made at or near the point at which the survey for a profile or section is begun, in order that this point may be recovered, in case a re-survey is required, or it is desired to connect the survey with that of some other line.

A hole drilled in a rock, a notch cut in the root or side of a tree, or any similar contrivance, may serve as a bench mark, the distance and bearing of which from the point where the survey begins should be noted.

The horizontal line to which the levels are referred in the survey is called the *datum* line.

In the preceding example, the horizontal line passing through the point at which the work begins, is taken for the datum line; in consequence of which some of the totals are positive and others negative. For greater convenience, it is usual to take the datum line at such a distance below the point of beginning as will render the totals all positive. This, it is evident, will not affect the relative value of the totals, since they are all merely increased by the same quantity.

Ex. Let the preceding example be plotted, assuming the datum line at 10 feet below the point of beginning.

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SECTION VII.

TOPOGRAPHICAL SURVEYING.

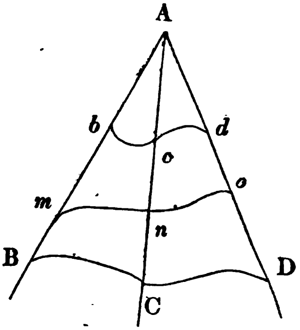
173. In addition to the particulars already noticed, the following objects may also be comprised in a survey;

1°. To determine the form or contour of the surface surveyed and to delineate it on paper, so as to represent to the eye the valleys, hills, and other inequalities of the surface. 2°. To describe the character of the surface, as covered with wood, or laid out in cultivated fields, orchards, gardens &c.; or as the site of buildings, villages, cities, &c. 3°. To indicate the nature of the structures erected upon it, as fences, bridges, mills, churches and other public buildings, &c.

That part of surveying which relates to these objects, is called *Topographical Surveying.*

174. The method for determining the form of the ground, its undulations and inequalities, consists, in general, in drawing on paper a series of curves representing the intersections with the surface of the ground of a series of horizontal planes taken at a given distance from each other. These curves, it is evident, will approach or recede from each other, as the ascent in any part is more or less steep.

Let it be proposed, for example, to determine the contour of a hill having its summit at A.



Driving a stake at A to determine this point, we mark down the sides of the hill as many lines AB, AC, AD, &c., as we judge necessary for our purpose. We then place the level at A, and put the target on the staff at a distance equal, in addition to the height of the instrument, to the distance we wish the secant planes to be from each other. The instrument being brought to a level, we next move the staff up or down one of these lines AB, for example, until, at the point *b*, the target is intersected by the horizontal thread of the telescope. Driving a stake at *b*, we next move the staff to the line AC, and determine in like manner, the point *c* in this line; and so on for the remaining lines AD, &c. The points *b*, *c*, *d*, &c., it is evident, will be on the same level; and the line which joins these will be the line in which the secant plane, at the proposed distance below A, comes out on the sides of the hill.

In like manner, we determine the points *m*, *n*, *o*, &c., at the same distance below *b*, *c*, *d*, &c., that these last are below A. We thus determine the intersections with the surface of as many secant planes, at the same distance from each other greater or less, as we deem sufficient, according as the hill is more or less steep, or as a greater or less degree of accuracy is required.

In this operation the instrument should be kept at A, until as many curves as possible are determined from that station. Let *m*, *n*, *o*, &c., be the points last determined from A; we move the instrument next to some position on, or a little above, the line which joins these points, from which to observe as many points as possible in the next line below. The place of the instrument may have to be changed several times before this next line is determined.

This being done, we next measure the distances A*b*, *bm*, A*c*, *cn*, &c., taking care to keep the chain horizontal; since it is the horizontal distances between these points that we want. We measure also the angles BAC, CAD, &c.; or, which will answer the same purpose, the lines *bc*, *cd*, &c.

In the present example let the following be the field notes.

Difference of level between the secant planes 8 feet.

Distances on the line AB, $Ab = 15$ feet
$bm = 21$ "
$mB = 10$ "

Distances on the line AC, $Ac = 14$ "
$cn = 12$ "
$nC = 17$ "

Distances on the line AD, $Ad = 14$ "
$do = 10$ "
$oD = 18$ "

Angles, BAC = 30°, CAD = 33°.

175. To plot the work, draw at pleasure a line AB on which lay off the distances A*b*, *bm*, *m*B, = 15, 21, and 10 feet respectively. Make next the angle BAC = 30°, through which draw AC; and on this line lay off the distances A*c*, *cn*, *n*C, = 14, 12, and 17 feet respectively, and proceed in like manner for the line AD. Draw through *b*, *c*, *d*, the curve *b c d*; and through *m*, *n*, *o*, the curve *m n o*, &c. The curves thus drawn will give an idea of the general form of the hill. Thus between B and *m* the ascent is very steep; between D and *o* it is much less so.

The general shape of the hill may also be made more apparent to the eye, by a proper shading of the parts between the curves.

This is done by filling up the spaces between the curves by lines perpendicular to them, and which will be shorter, nearer, and heavier, in proportion as the hill is more steep, and the converse.

The difference of level between the planes, and the distances measured, furnish data for drawing profiles of the hill in the direction of the lines AB, AC, &c., by which its form may be still further exhibited.

Ex. To determine the contour of a hill from the following notes.

AB	AC	AD	AE	AF	AG	AH
10	7	6	8	15	5	8
5	6	6	12	12	8	4
22	17	12	10	7	8	11
35	12	10	19	10	9	11
29	16	19	11	9	11	12
27	17	10	8	6	12	5

AB, AC, &c. denote severally the lines drawn from the vertex A down the sides of the hill. The numbers beneath them denote respectively the distances from each other of the points where the secant planes come out upon these lines. The following are the angles contained between the lines.

BAC = 24°
CAD = 40°
DAE = 69°
EAF = 48°
FAG = 60°
GAH = 74°
HAB = 45°

176. We have found the points A, *b*, *m*, &c., so that the differences of level between them shall be equal to the assumed distance between the secant planes. This is the most convenient for plotting, though not the most convenient for the operations in the field. In general, therefore, the following course may be pursued.

Along the line AB drive stakes at *b*, *m*, B, &c. as is found most convenient, or as we judge will best subserve our purpose, and take the differences of level between these points, and measure also the distances A*b*, *bm*, &c. as before. We then regard the differences of level as uniform for the distances A*b*, *bm*, &c.; and having assumed a convenient distance for the distance between the secant planes, we determine by a simple proportion the points where these planes will come out on the line AB. We proceed in like manner upon the remaining lines AC, AD, &c., and then plot the work as before.

Suppose, for example, that A*b* is 12 feet, the difference of level between A and *b* is 8 feet, and that we wish the secant planes to be at the distance of 6 feet from each other. The first secant plane will then be 2 feet above the plane passing through *b*; and to find where on A*b* the secant plane cuts the ground, we have

$$8 : 12 :: 2 : 3.$$

The secant plane cuts the ground, therefore, at 3 feet above *b*, or 9 feet below A. The proportions required in this case are performed practically with great facility by aid of the sector.

We take next the following example, viz.

Ex. To trace the passage of a river through the notch of two mountains.

To trace the river, beginning at a station 0 on the west side, we take the following bearings, distances, and offsets to the line of the shore.

Sta.	Bearing.	Dist.	Offsets.	
N.		Ch.	L.	R.
		5.00		.30
		1.40	.15	
M.	N. 30° E.	0.30		.10
		3.00		.40
		2.00	.00	.00
C.	N. 50° E.	0.50		.15
		3.00		.20
		2.20	.15	
		1.30		.20
		.80	.00	.00
B.	N. 20° E.	0.20		.07
		4.40		.20
		2.00		.40
0.	N. 10° E.	0.00	.00	.00

In the column of distances, the distances of the offsets from the beginning of the course are written in order from the bottom upward, the number last written being the whole length of the course or distance. Thus 4.40 is the first distance, 3 the second and so on.

The breadth of the river at 0 is 1.80, at M .60, and at N .50 chains. From B to station *b*, at the water's edge, on the opposite shore is 1.20 chains; from C to *c* do. 1.05 chains; each on a perpendicular to BC.

To determine the vertices A, *a* of the mountains, we have, on the west, the base BC = 3 chains, and the lines AB, AC equal 2.90, and 2.65 chains respectively; and, on the east, the base *bc*, and the lines *ab*, *ac* equal 3.05, and 3.45 chains respectively.

To determine the form of the mountains, the lines AB, AC, *ab*, *ac*, &c. are laid off from the summits A, *a*, respectively, down the sides. The secant planes are taken at a distance of 10 links, or feet, from each other. Passing from the top downward, the distances between the points at which the secant planes come out upon the lines are noted in the following tables.

West Mountain.

AB	AC	AD	AE	AF	AG	AH
.20	.15	.20	.15	.18	.20	.30
.30	.35	.30	.25	.20	.30	.40
.60	.75	.80	.60	.55	.50	.60
.50	.40	.65	.50	.40	.40	.80
.40	.20	.80	.70	.50	.35	.90
.90	.80	.95	.60	.60	.80	1.25

East Mountain.

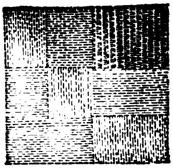
ab	*ac*	*ad*	*ae*	*af*	*ag*	*ah*
.30	.25	.19	.25	.20	.25	.40
.45	.40	.40	.50	.20	.30	.25
.60	.75	.80	.80	.40	.50	.80
.70	.50	.50	.40	.50	.20	.30
.45	.85	.90	.60	.65	.60	.95
.25	.30	.70	.75	.30	.50	1.00
.30	.40	.55	.80	.40	.30	.50

The angles between the lines AC, AD, &c. are laid off from AC through the north and west round to AB; and are as follows, viz. $CAD = 41^\circ$, $DAE = 44^\circ$, $EAF = 60^\circ$, $FAG = 45^\circ$, $GAH = 60^\circ$.

The angles between the lines *ac*, *ad*, &c., are laid off from *ac* through the north and east round to *ab*; and are as follows, $cad = 40^\circ$, $dae = 43^\circ$, $eaf = 42^\circ$, $fag = 50^\circ$, $gah = 61^\circ$.

177. To indicate the nature of the surface, certain characters, or conventional signs have been adopted by the U. S. Topographical Bureau, and are used in all plans and maps made by the U. S. Engineers.

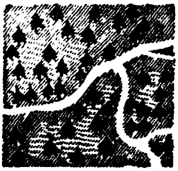
Fig. 1.



Cultivated ground is represented by fine interstitial, or dotted lines, parallel to each other, as in figure (1).

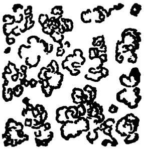
Gardens are represented as cultivated ground, but laid out in a more ornamental manner with walks, trees, &c.

Fig. 2.



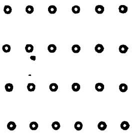
Wood lands are represented by trees drawn as they appear standing; the different forms of the trees indicating the different kinds of wood land. Thus figure (2) represents a pine wood.

Fig. 3.



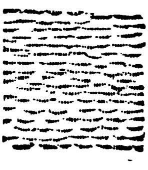
It is in general, however, most convenient to represent wood lands by trees as they would appear in projection upon the ground, as in figure (3), which represents an oak wood.

Fig. 4.



Orchards are represented by trees in parallel rows, or by very minute circles arranged in the same manner, and indicating the projections of the trees on the ground as in figure (4).

Fig. 5.



Grass ground is indicated by clusters of short vertical lines or strokes of the pen, representing tufts of grass, and arranged uniformly over the surface.

Pasture is represented by the same kind of clusters scattered unequally over the surface as in figure (5).

Flowing water, as rivers, brooks, &c., is represented by curvilinear or wavy lines drawn parallel to the shore. The lines are drawn fainter and the intervals between them increase as they recede from the shore. A brook is represented in this manner in figure (8).

Standing water, as lakes, ponds or pools, is represented by fine horizontal lines drawn from left to right either entirely across the surface, or for a short distance around the shore.

Fresh marsh is represented by pools of water interspersed with clumps of grass as in figure (6).

Salt marsh is represented by horizontal lines with scattered tufts of grass as in figure (7).

Meadows are represented as in figure (8).

Fig. 6.

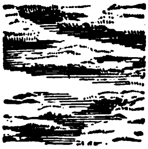


Fig. 7.

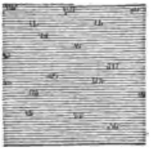
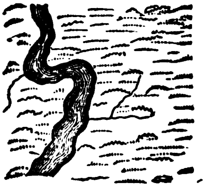


Fig. 8.

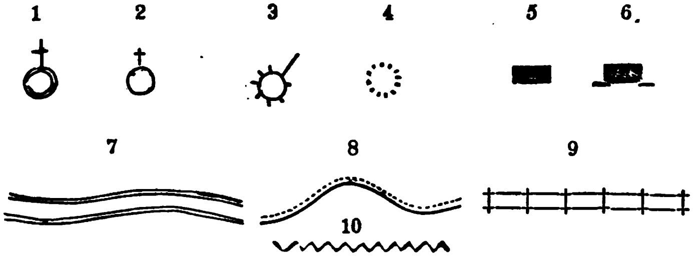


Heath is represented by tufts of grass interspersed with bushes and shrubs.

Deep morass is represented in the same manner as fresh marsh, with the addition of a few bushes and clusters of short irregular vertical strokes to represent flags.

Sand is represented by dots. Thus a sandy shore is represented by dotted lines parallel to the line of the shore.

178. Signs are also adopted to represent the structures and other artificial objects upon the surface. Thus of the following figures, the first represents a village church, the second a church detached; the third represents a saw and the fourth a grist mill; the fifth represents a house, the sixth a tavern; the seventh represents a common road, the eighth a foot path beneath which is the sign for a wooden fence; the ninth represents a railroad.



A court-house is represented the same as a house, with the addition of a small pair of scales upon the upper side of the figure. A post-office is also represented in like manner, with the addition of a small trumpet upon the upper side. A cemetery is represented by a small grave stone with a cross upon its face.

Signs are also adopted for other objects. A map in which the topography is carefully executed, in the manner which has now been explained, conveys at once to an instructed eye the same information which would be obtained by actual inspection of the ground itself.

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SECTION VIII.

SURVEYS FOR RAILROADS, CANALS, &C.

179. The leading objects embraced in a survey for a railroad, are, 1°, to make a map of the route including such contiguous objects as it may be of importance to notice; 2°, to make a plot or

profile of the line of the road; 3°, to establish the grades, or inclinations of the line, required in passing from point to point.

The best line for the direction of a railroad is the straight line joining its two extremities. This, however, is seldom feasible, except for very short distances. In general we are obliged to follow the courses of streams, to wind round among valleys, &c., in order to avoid high hills and other obstructions.

Map of the Survey.

180. The surveyor will first pass over the region through which the road is to be made, selecting, so far as the eye can judge, the best route between the extreme points. This is called a *reconnaissance.*

The leading or guiding points of the line being thus established, signals are next placed along the route, at every point where a change is made in the course. The bearings from each other of the stations thus marked, are then taken, and the distances between them measured. The bearings also of such points or objects on either side of the route, as it is desired to notice in the plan, are also taken; or, as may be more convenient, their positions are determined by means of offsets.

The character of the ground, or its *topography*, is also sketched at the same time—that is, the general nature of the surface, whether it is covered with wood, or is cultivated; whether it is meadow or high land, or rocky, marshy, &c. The signs by which these circumstances are represented to the eye, have already been explained.

181. The field book is usually kept in the following manner.

The book is ruled across with parallel lines, at a convenient distance from each other, to represent a chain, or any number or parts of a chain at pleasure. The left hand page is ruled, up and down, in convenient spaces to write down the number of the station, the bearing and distances of the stations, &c. The right hand page is divided into two parts at the middle by a single vertical line.

A 0 is used to indicate a station point. Beginning at the bottom of the page a 0 is placed at the intersection of the vertical with the bottom line; and against it, on the same line in the left hand page, the number, bearing and length of the first course is entered. Let the length of this course be 5 chains, for example. We take 5 of the horizontal spaces to represent this distance; and placing a 0 on the vertical line of the right hand page, at the fifth line from the bottom one, to indicate the second station point, against this, on the same line on the left hand page, we enter, as before, the number, bearing, and length of the second course; and so on.

The topography is then sketched, by aid of the proper signs, on either side of each course as we proceed. In like manner the forms and relative position of objects of minor importance may be noted. Or, when accuracy is required, their position as determined by bearings or offsets may be noted.

The distance of an offset from a station on the line of the course is entered in the column of distances on the left hand page. The perpendicular distance is entered on the right hand page, beneath a perpendicular line, drawn from the line of the course to the point or object determined by the offset.

Thus, between stations 1 and 2 the line of the route may pass through cultivated fields; between 2 and 3 through a grove of oaks or other trees; between 3 and 5 along the margin of a brook, with meadow or marshy ground on either side. These should all be properly sketched in the field book. The margin of the brook may be traced sufficiently near for all practical purposes by the eye, by observing carefully its windings in relation to the line of the course near it. Or, if greater accuracy is required, offsets to it may be made to as many points as we please.

The points at which the line crosses roads, creeks, brooks, &c., should be carefully noted and entered in the field book.

In this manner, it is obvious, a very perfect map may be made of the whole route, giving at each point all the information necessary for the purposes of the survey.

182. Ex. Let the learner now draw a plan of a route for a railroad, with its topography from the following field notes.

The route commences at the outlet of Lake Clear and extends to Broad Bay, by the following courses and distances.

Sta.	Bearing.	Dist.
6	N. 63° E.	30.75 ch.
5	N. 32° E.	26.40
4	N. 20° W.	30.00
3	N. 5° E.	20.00
2	N. 25° E.	26.50
1	N. 45° E.	22.30
0	N. 60° E.	25.00

NOTES.

1. From 0 station to 2, the line is on the westerly side of a road, one chain in width, leading from the outlet of the lake to the bay.

2. From 0 to 1 the line is also on the right margin of a brook, which, proceeding from the lake, makes a turn northerly at 1 and is again met by the line at 3. From this station the brook proceeds in a direction S. 80° E. At 2 an offset to the brook was found to be 11 chains; and at 13 chains from 2 an offset is 9 chs.

3. From 2 the road from the lake runs N. 47° E. to its termination at the bay.

4. On the right of the road from 0 to 1 is a thick pine grove; from 1 to 2 is pasture land; and from 2 to the intersection of the road with the brook, is a cultivated field. On the left between the road and the brook is meadow land.

5. From 0 to 3 on the left of the brook, is a pine wood. From 3 to 4 the line passes over a fresh marsh. From 4 the ground rises gradually to 5, and is cultivated on both sides of the line.

6. On the left from 5 to 6 is a grove of oaks. At 6 the line meets the right bank of a river which, coming down from the north

west, makes a sudden bend at this point. From 6 to its termination at 7 the line is on the bank of the river. At 6 the river is 2 chains in width and at 7 it is 3 chains. Commencing at the point opposite 7, the following courses were run along the line of the river and bay, viz; 1, N. 10° E. 15 chains; 2, N. 24° E. 21 chains.

7. Through 6 passes the westerly line of a road, intersecting the lake road to the bay at right angles. From the point of intersection to the bay is 40 chains. And at 25 chains, toward 6, from the point of intersection, an offset to the bay is 20 chains. The ground on both sides of the cross road is occupied by dwelling houses, gardens, orchards, &c.

8. At station 4 the summit of a high hill was observed bearing N. 70° W. At 5 the bearing of the hill is S. 45° W. At 5 also the bearing of another hill is N. 40° W. At 6 the bearing of the same hill is N. 87° W.

The above are designed as general directions only; and may, with the courses and distances be varied at pleasure. As an additional exercise, let the learner prepare a field book, sketching the different objects he is supposed to meet, and making the requisite entries in the manner explained above; and then plot the work from his field book.

Profile of the Route.

183. Upon the whole line of the route, as thus determined by the chain and compass, stakes, beginning at the point of departure, are driven into the ground at the distance of one chain, or 100 feet, from each other, to serve as stations for the levels. They are numbered, beginning at the first, 0, 1, 2, 3, &c. Thus the number of the station will give its distance in chains, or multiplied by 100 its distance in feet, from the first station, or point from which we started.

If intermediate stations are required, as in crossing a ravine, for example, pins are driven into the ground upon the edge of the banks, bottom, &c., and their distance from the primary station next preceding marked upon them.

The stations being thus prepared, the next operation is with the level.

To save labor the instrument should be placed successively in positions to observe, at the same time, as many of the stations as possible; taking care always to place it as near the centre of the extreme stations observed as practicable, in order to avoid the corrections for level.

The observation upon the staff at the 0 station, or point of departure, is the *first* back sight; all others are regarded as fore sights in whatever direction the instrument is turned. The station upon which the last fore sight is taken, becomes the station for the *second* back sight and so on. The differences between each sight and the one next following are the differences of level, which are marked + or —, according as the ground is ascending or descending.

The stations or points at which the level is moved, are called *Turning Points*. Observations upon these should be made with great care; as any error upon them will affect the whole subsequent work. Errors between the turning points balance each other.

When the datum line passes through the 0 station, the difference between the sum of the fore sights and the sum of the back sights should be equal to the last total. This furnishes a test of the accuracy of the work. If the datum line is taken below the 0 station, then the difference between the sum of the fore and back sights should be equal to the difference between the last total and the assumed distance of the datum line below the 0 station, or the difference between the first and last total. In applying this principle, the fore sights on the turning points and the last fore sight only must be reckoned. Subjected to this test, the work in the following example, will be found to be correct.

184. In the survey of a route for a railroad the following record of the levels was kept. It is required to plot the profile.

Station.	B. S.	F. S.	Diff.	Total.	Remarks.
0				12.00	
B.M.	8.00	6.00	2.00	14.00	Bench mark on an oak tree 20 ft. left of 0 + 20
1		4.00	2.00	16.00	
.20		5.25	— 1.25	14.75	
.30		9.75	— 4.50	10.25	
2		2.00	7.75	18.00	
.50		.00	2.00	20.00	
3 Pt.		4.00	— 4.00	16.00	
.30	4.30	1.05	3.25	19.25	
.70		13.80	—12.75	6.50	
4		2.30	11.50	18.00	
5		5.70	— 3.40	14.60	
6		3.55	2.15	16.75	
Pt.		3.30	.25	17.00	On rock to the left.
7.	1.63	5.13	— 3.50	13.50	
8		9.81	— 4.68	8.82	
9 Pt.		15 06	— 5.25	3.57	
10	5.06	6.13	— 1.07	2.50	
11		2.63	3.50	6.00	
12		2.13	.50	6.50	
B.M.		1.50	.63	7.13	Elm tree 30ft. right.

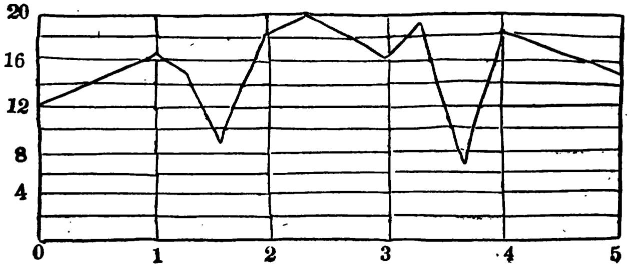
The back sights are written one line below the stations or points to which they refer. Thus the back sight 8, written on the same line with the point B.M., is the back sight taken upon the station 0; and so on. In the difference column the difference first written, in each position of the instrument, is the difference between the back sight and first fore sight; and next the differences between the fore sights in succession. These differences take the positive or negative sign, according as the one that follows is less or greater than that which immediately precedes it; or, as the ground is ascending or descending.

The totals give the height of each point above the datum line. This is assumed at 12 feet below the 0 station, which, in the present example, is sufficient to render the totals all positive. The totals for points without the line, as those for the bench marks and the turning point between stations 6 and 7 in the present example, should not be plotted with the rest.

From the field book it appears, that bench marks are placed near each extremity of the line. In general, bench marks should be placed along the route at such points as convenience, or the final location and construction of the road may require.

185. To facilitate the plotting of the levels, ruled paper is prepared, with parallel lines at a convenient distance to represent feet. These are intersected by perpendicular lines, at a distance from each other equal to five times that of the former. If the distances between the perpendicular lines are taken for the distances between the stations, or regarded as 100 feet each, the plot will be made on a scale of one foot in the difference of the levels to 20 between the stations; the scale, for the reasons already explained, usually employed.

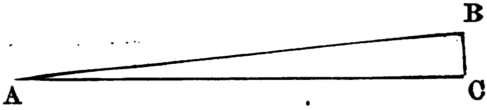
The course to be pursued is illustrated in the following figure, in which the plotting of the example is carried to the 5th station. The differences of level are plotted on a scale of 20 feet to the inch, and the distances between the stations on a scale of 200 feet to the inch.



The datum line in the present example is taken, as we have remarked, at 12 feet below the 0 station. In surveys along the seacoast or extending from it, it is usual to take the datum line at 25 feet below high water mark. This is, in general, found sufficient to render the totals all positive, and to prevent the work from running below the ruled paper in the process of plotting.

Establishing the grade.

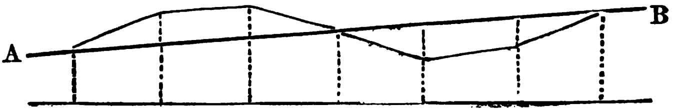
186. By the grade is understood the longitudinal slope of the road or its inclination to a horizontal plane. The grade is measured by the proportion between the difference of level of the extreme points of the line and its horizontal length or projection on a horizontal plane.



Thus, let A and B be the extreme points of the line, BC the difference of level between these points, and AC the horizontal length of the line, or its length as actually measured. The grade or inclination of the line will depend, it is evident, upon the ratio of BC to AC. Let BC = 1 foot, AC = 100 feet; the grade is then one foot to a hundred, or 52.8 feet to the mile. This is the highest grade which, in general, should be admitted in the construction of the road.

187. The profile of the route being drawn, we proceed next to establish upon it the *grade line.* This, it is evident, must follow the general direction of the surface. It must also be determined in such a manner, that the grade being kept within the required limit, the minor elevations above the line shall be equal, or nearly so, to the depressions below; so that the cut in the protuberances above the line will fill the hollows below with the least possible moving of the earth.

Let the following figure be a profile of a section of the road.



We draw an experimental line AB, the inclination of which is within the prescribed limit. If then, on calculation, the cutting is found to be equal, or nearly so, to the filling, AB will be the grade line required; otherwise it must be varied until the required conditions are fulfilled.

If on passing over a hill the grade is found to be greater than is admissible, the hill must be cut down, or the line of the road be made to wind along the face of the hill in such a manner that its length will be increased, and thus the grade be diminished.

Excavation and Embankment.

188. The line to which we have referred, is that which passes through the centre of the road, the profile being drawn upon the vertical plane which passes through it.

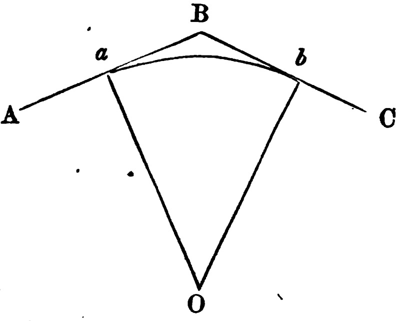
To determine the cuttings and fillings, cross sections of the road will be required at the stations, and occasionally at intermediate points. The levels for these should be taken in the survey, and properly entered in the field book. Supposing a cut between two stations, the sections of the cut at the stations will form the bases, and the distance between the stations the length, of the prismoidal figure, the content of which is to be calculated in order to determine the quantity of earth to be removed. The area of the cross station will depend, it is evident, upon the breadth of the road, the inclination or slope of the sides and the depth of the cut. In common soils the slope is usually taken at 1½ feet out to 1 foot elevation; that is, the slope is the hypothenuse of a right angled triangle in which the base is to the perpendicular as 1½ to 1,

A section of a cut is represented in the figure. The depth of the cut is 16 feet, the breadth of the road 40 feet, This gives 88 feet for the upper line, the slopes being made in the proportion of 1½ to 1, A section for an embankment is the same with that for a cut inverted, with the exception that the slope is made in the proportion of 2 feet out to 1 foot elevation.

In the preceding example, the surface of the ground is supposed to be level at the cross sections. This is rarely the case, however; and the figure of the section will vary accordingly.

Curves.

189. Thus far the turns of the road are supposed to be at angles more or less obtuse. This is inadmissible in the actual construction of the railroad; and the turns must be made on curves uniting the lines which form an angle in such a manner, that these lines will both be tangents to them. The most simple curve for this purpose is the circle; the radius of which should be taken as large as possible. In no case should it be less than 1000 feet.



Let two lines of the route, AB, BC meet at the point B; it is required to mark on the ground the arc of the circle by which they may be united.

Let a on the line AB be taken for the point at which it is desired the curve shall commence. On BC set off Bb equal to Ba; and at the points a and b erect the perpendiculars aO, bO to AB, BC respectively. The point O where these perpendiculars meet will be the centre of the required curve; which may now be described with the radius Oa or Ob. We shall have then for the line of the road between A and C the line A$ab$C.

On account of the length of the radius, this method, which is very simple, cannot often be employed in practice. Methods are, therefore, devised for laying off an arc of any radius without the necessity of finding its centre. Instead of the circle other curves, especially the parabola, may be employed to unite two lines of the road which make an angle with each other.

COMMON ROADS.

190. The survey for a common road embraces several of the particulars required in that for the railroad. Higher grades are, however, admissible; and in order to a proper drainage, the road should never be made perfectly level. The proper limits for the grade of a common road are 1 in 125 for the least, and 1 in 30 for the highest grade.

The common roads of this country have, in general, been badly laid out, though now becoming improved in consequence of the information diffused in the construction of railroads.

Upon the line of an old road the following levels were taken; it is required to show how the road may be improved by suitable grades, the earth from the cuts being moved as little as possible in filling the hollows.

Station.	B. S.	F. S.	Diff.	Total.
0				20.00
1	7.00	1.00	6.00	26.00
2		9.00	— 8.00	18.00
3.20 Pt		3.00	6.00	24.00
4 Pt.	10.00	.00	10.00	34.00
5	8.00	2.00	6.00	40.00
6 Pt.		12.00	— 10.00	30.00
7	1.00	15.00	— 14.00	16.00
8 Pt.		5.00	10.00	26.00
9 Pt.	10.00	2.00	8.00	34.00
10	8.00	1.00	7.00	41.00
11		4.00	— 3.00	38.00
12		11.00	— 7.00	31.00
13 Pt.		7.00	4.00	35.00
14	1.00	12.00	— 11.00	24.00

CANALS.

191. The survey for a canal route does not differ materially from that for the railroad. In the case of the former, however, a

greater deviation from the straight line which joins its extremities may be necessary, in order to secure a more level route, or the necessary supply of water to feed the canal. In the case of the canal, moreover, the line does not admit of being inclined in any part to the horizon; and the passage from one level to another as may be required, is effected in a vertical direction by means of a contrivance called a lock.

192. We close with the following example.

Ex. A small stream passes along the bottom of a ravine. It is required to make a section of the ravine from the following notes, and to show what head of water will be obtained by throwing a dam across it to a level with the right bank.

Station.	B. S.	H. I.	F. S.		Remarks.
B.M.				30.00	Nail in stump.
0.	7.24	37.24	4.25	32.99	Right bank.
.30 Pt.			8.48	28.76	
.70 Pt.	0.75	29.51	9.35	20.16	
1.43	0.32	20.48	5.97	14.51	Bottom of slope.
.00			8.34	12.14	Level of stream.
.70			6.98	13.50	Bottom 2d slope.
.80 Pt.			0.31	20.17	
2.00 Pt.	10.62	30.79	1.80	28.99	
.20	6.53	35.52	1.52	34.00	Left bank.

In this example we have another method of keeping a record of the levels.

The column marked H.I. contains the height of the line of sight of the instrument above the datum line. The bench mark is assumed at 30 above the datum line, and the back sight on this being 7.24 the first height of the instrument is 37.24. The first fore-sight is 4.25; the 0 station is, therefore, 4.25 lower than the height of the instrument or 32.99; and 0.30 is 8.48 lower or 28.76. At 0.30 the instrument is moved; and the back sight being 0.75 gives 29.51 for the second height of the instrument, and so on. In general the height of the instrument is found by adding the back sight to the height of the point on which it is taken, or the last total. And the height of the successive points are found by subtracting the foresights from the height of the instrument.

The method is called the Height of Instrument method. It is more expeditious than the preceding, or the + and — method; but there is greater liability to error in its use. It is not, therefore, so safe for beginners.

SECTION IX.

TRIANGULATION. MORE EXTENSIVE SURVEYS.

193. The ordinary methods of land surveying being now understood, we pass to more extensive surveys; such as are required for the construction of maps of counties, the smaller states, and other similar portions of the earth's surface.

Surveys of this description may be divided into two distinct parts; 1°. To determine with precision the position of the most prominent points of the territory to be surveyed, by means of a chain of triangles having these points for their vertices and extending over the whole surface. 2d. To fill in, by the usual processes of surveying, the rivers, roads, villages, bounding lines, and other characteristics of the surface.

Reconnaissance.

194. A reconnaissance of the territory is first made in order to determine the location most suitable for the measurement of a base line, and the leading points to be connected by the series of triangles.

Let the following figure represent the territory it is required to survey. Let it be supposed that a base line may be measured between the points A and B, and that A, B, C, D, E &c. are the more prominent points of the surface. Conceiving lines to be drawn uniting these points, we shall have the chain of triangles required.

Measurement of the base line, and angles.

195. Proper signals having been placed at the points A, B, C, &c. the first step is to measure the base line. This should be done with the greatest care, as the accuracy of the entire work depends upon it. Having measured the base, we proceed next to measure the angles in the series of triangles. The instrument for this purpose is the theodolite, the use of which has already been explained.

Placing the instrument at A, one extremity of the base line, for example, we measure the angles CAD, DAB, BAK, &c. of the triangles CAD, &c. which have one of their angles at this point. Removing next the instrument to B we measure, in like manner, the angles DBA, DBE, &c. We proceed in the same way at all of the other stations C, D, E, &c., until all the angles in the several triangles of the series are measured.

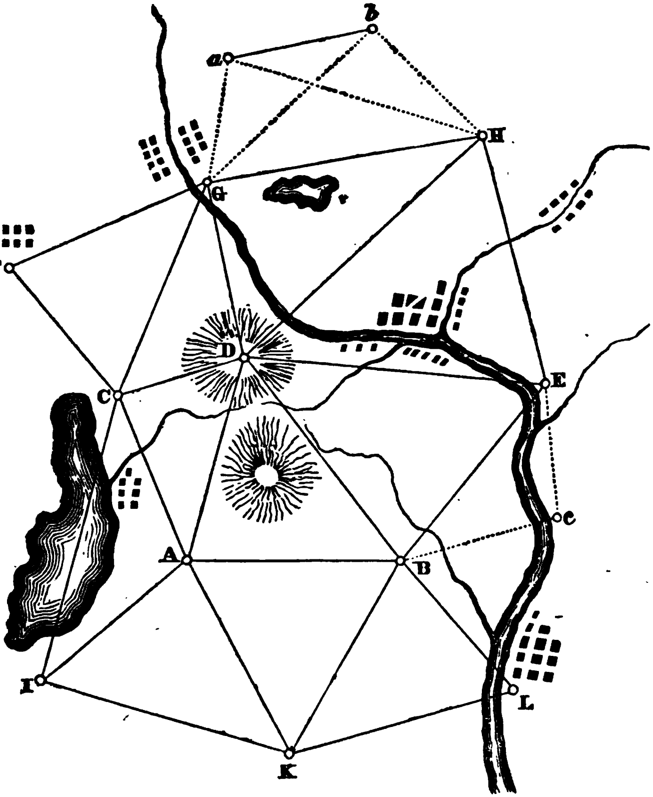
196. The process of determining the position of a series of points by means of a system of triangles by which they are connected, is called a *triangulation.*

The process of determining the principal or primary triangles of the series, is called the *primary triangulation.*

From the sides of the primary triangles, as bases, a second series of smaller triangles, comprehended within them, may be formed. These are called *secondary* triangles; and the process by which they are determined is called the *secondary triangulation.*

If the secondary triangulation does not furnish a sufficient number of points conveniently situated for filling in the details of the survey, other triangles still smaller may be formed from the sides of the secondary triangles, constituting a third or *tertiary triangulation*; and so on.

The triangles of the entire system are called triangles of the *first*, *second*, *third order*, &c., according to their relative importance in the system.



In the figure the triangles ABD, DBE, &c. are the primary triangles. The triangle B*c*E is a secondary triangle formed from the side BE of the primary triangle BDE, for the purpose of determining the point *c*.

All the angles in each of the primary and secondary triangles should be measured. The sum in each case compared with two right angles, will test the accuracy of the measurement. If the error exceeds in any case the limit prescribed, the angles should be measured again.

In triangles of the third order, or those designed to establish single points only, the third angle may be inferred from the other two. But, in general, all the angles of the triangles employed should, if practicable, be measured.

As a test of the general accuracy of the work, an additional base line should be measured, at a distance from the first. The length of this being calculated from the first, the result, it is evident, should agree with its length obtained by the measurement. Any material disagreement between the two will require a re-examination of the work until the source of the error is discovered.

In the figure a second base line *ab* is measured, and connected with the side GH of the main triangulation. The base line AB is called the *primary base*; and *ab* the *base of verification.*

Filling in the Survey.

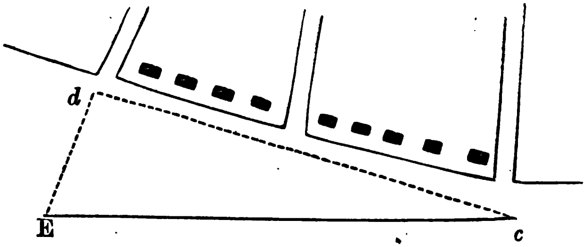
197. A sufficient number of points being determined, with the greatest possible accuracy, by the triangulation, we proceed next to the detail surveys. For this purpose it will be convenient to divide the territory into sections of moderate size. And, as in the present example the town lines are to be drawn upon the map, we will take these for the boundaries of the sections.

Let it then be supposed that the position of two, or a greater number of points, are determined in each town by the triangulation.

The lines or sides of the triangles joining these, already known by calculation, will form so many base lines from which the survey of the sections may proceed, according to the methods already explained.

Thus from the side *c*E of the secondary triangle B*c*E, offsets may be made to the river, by which its course near this line will be determined.

Proceeding from *c*E, we set up next a staff at *d*, as in the following figure, and measure the lines E*d*, *cd*. By means of offsets from the line *cd*, we determine the adjacent objects, as the principal street of a village, for example, and the houses, gardens, public buildings, &c., situated upon it. We proceed in like manner with the line E*d*.



The lines *cd*, E*d*, again, will furnish points for new stations; the lines joining which may be measured and offsets from them to the objects near may be made; and so on, until the survey of the entire section or town is completed.

Plotting the work.

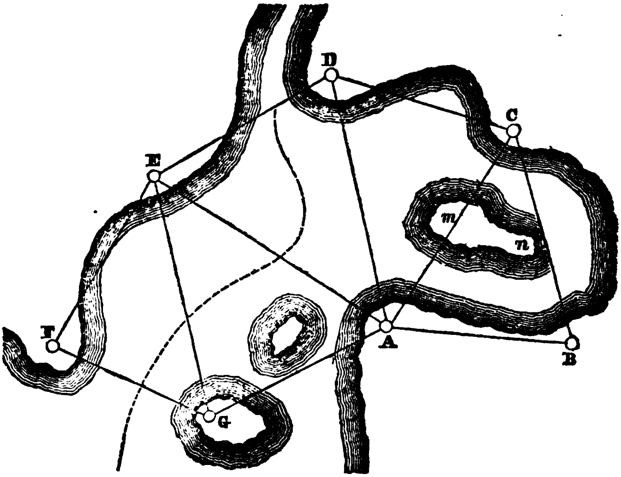
198. To determine the position of the territory in respect to the points of the compass, a meridian line should be determined at some point, the extremity A of the base line, for example. The bearing of any one of the sides proceeding from this point being taken, the required position of the territory will then be known.

Laying down on paper the base line AB, the several triangles may be plotted from this by means of the angles which are measured, and thus a plot of the triangulation be made. Or, assuming the meridian passing through A and the perpendicular upon it drawn through this point as axes, we may calculate, with reference to them, the co-ordinates of each point of the triangulation, as explained in art. 145; and then plot the triangulation by means of these co-ordinates.

The sectional surveys being all plotted, in the usual manner, to the same scale, may all be united by means of the points of the triangulation upon which they depend; and thus the map of the territory be completed.

199. The territory surveyed may comprehend a portion of the sea-coast, with its inlets, projections, harbors, &c. which will require some additional operations.

Let the following figure represent a harbor forming a part of the survey.



The survey of the harbor should embrace the following particulars; 1°. To trace the shore along the line of high water mark, taking care to notice all the prominent points such as inlets, places where streams discharge themselves into it, &c. 2°. To determine the depth of the water; to mark the channel; and to define the position of rocks, shoals, &c., or whatever may affect navigation. This part of a survey is called *hydrography*.

Let A, B, C, D, &c. be the more prominent points along the shore, which we suppose to be determined by the secondary triangles ABC, ACD, &c. forming a part of the general triangulation.

To trace the shore line, we will employ the chain and compass. Commencing at F, for example, we run lines as near the water's edge as possible, following all the bends and indentations, and making offsets when necessary for a more accurate delineation of the line of the shore. Between the points thus determined, a sketch of the sinuosities of the shore line may be made in the field book in the manner explained for the chain and compass work in leveling.

The chain and compass work should always connect with the vertices of the principal triangles, so that any small errors in the work between any two of these points may not be propagated beyond them.

To determine the position of other points, or objects, we may lay off from the sides of the triangles already formed as many other triangles as we please. Thus two points m and n on the island opposite AB, may be determined by triangles of the third order, ABm, ABn. These points being fixed, the shore line of the island may then be easily traced with the chain and compass.

200. To determine the depth of water various methods are employed.

1. In lines at suitable distances from each other, buoys may be anchored across the harbor, and their positions determined by their bearings from the vertices of the triangles or other fixed points. The depth of the water may then be taken at the buoys.

2. Instead of the buoys a boat may be anchored successively at different points along the lines of the soundings, and its position in each case being determined, the depth of the water may be taken as at the buoys.

3. Or on the lines on which soundings are to be taken, a boat may be rowed with a slow and uniform motion, and the depth of the water taken at equal intervals of time. This is, however, a

rough and approximate method, and the former methods are preferable.

The depth of water at the soundings being marked upon the plan, a dotted line drawn through the points of deepest water will indicate the channel.

By means of the soundings reduced all to the same level, that of high or low water mark, a complete map of a shoal, or the entire bottom of the harbor may be delineated, in the same manner as already explained for the contour of a hill.

PLANE TABLE.

201. A very convenient instrument for mapping the details of a survey is the plane table, the construction and use of which we will briefly explain.

The principal parts of the plane table are a board about 16 inches square and a tripod for its support.

The table is bordered by brass plates about an inch in width, and its centre is marked by a pin. By means of lines drawn from the centre, the outer edges of the plates are graduated so as to represent degrees, half degrees, &c. The degrees are numbered from left to right from 0 to 180°. The 0 is placed at one extremity of a diameter drawn through the centre, parallel to one of the sides, and 180° at the other extremity.

Two brass plates are fitted into grooves on opposite sides of the board, which, by means of screws from beneath, may be raised so as to permit a sheet of paper to be laid upon the board with its edges beneath them. The plates then being screwed down, the paper is drawn tight across the board and thus secured to it.

The table is attached to the tripod by a ball and socket joint, with brass plates and screws for the purpose of leveling it. This is done by aid of a detached spirit level.

Finally the instrument is fitted with a movable ruler about the length of the diagonal of the board, with sight vanes like those of the compass screwed on to each end. The fiducial edge of the ruler is in the same plane with the vertical threads and slits of the sight vanes.

The plane table may be used for the measurement of horizontal angles. Its principal use, however, is for determining the shorter lines of a survey both in extent and position.

Thus, let it be supposed that we wish to trace, from the base line AB, the southern shore of the island *mn* of the preceding figure.

We place the instrument at A, with a sheet of paper upon it, so that the larger part of the paper shall be toward the shore, and bring the instrument to a level. By means of a plumb line we then stick a pin in the board directly over A. Placing the edge of the ruler against the pin, we sight to the end B of the base line, and draw a pencil line along the edge of the ruler toward B. Suppose a series of stakes driven along the shore and numbered 1, 2, 3, 4, &c. We place the edge of the ruler against the pin, and sight in succession to the stakes, and draw lines in pencil indefinitely toward them, numbering them 1, 2, 3, 4, &c. We next lay off, by means of a convenient scale of equal parts, on the line drawn toward B, a line equal in length to the base AB, and place a pin at the end. We then remove the instrument to the station B, and bring it to a level, with the pin last stuck in directly over B; and, by means of the ruler, we arrange the line between the two pins on the board in the direction of the base BA. We next place the edge of the ruler against the pin over B, and sighting back to the stakes 1, 2, 3, &c. we draw lines toward them intersecting those before drawn. The points of intersection will be the place of the stakes as represented on the board; and, connecting these points by lines, we shall have a trace of the shore upon the scale used in laying down the distance between the two pins, or base line on the board corresponding to the base line AB.

202. The following example will serve as an illustration of an extended survey. The territory is situated on the sea-coast. A river with a tributary from the west, passes in a southerly direction nearly through the centre of it.

The following table exhibits the primary triangulation. The angles in the several triangles are all measured. Assuming the point A as the first station, the side AB of the triangle ABC is computed from a base line measured in its vicinity. The remaining sides of the triangles are all computed from this and their values written in the table.

PRIMARY TRIANGULATION.

Triangles.	Angles.	Sides.	M. D.	P. D.	Pt.
I.	BAC 56°	AB 387.72	—376.01	—100.35	B
ABC	ABC 59°	AC 409.95	—309.30	268.99	C
	ACB 65°	BC 375.00			
II.	CDA 59°	AD 444.98	27.16	444.15	D
ACD	ACD 68° 30′	CD 379.43			
III.	DAE 58°	AE 453.68	398.70	216.47	E
ADE	AED 60°	DE 435.74			
IV.	EAF 53° 30′	AF 374.43	339.35	—158.24	F
AEF	AEF 52° 30′	EF 379.39			
V.	FEG 51°	FG 342.19	674.67	— 90.01	G
EFG	EGF 69° 30′	EG 412.43			
VI.	HFG 66° 30′	FH 336.36	532.29	—433.79	H
FGH	FGH 56°	GH 371.26			
VII.	AFI 49° 59′.3	FI 535.88	—146.35	—384.61	I
FAI	FAI 85° 50′	AI 410.02			
VIII.	AIB 59° 35′	BI 364.55			
AIB	BAI 54° 10′				
IX.	KDE 64° 07′	DK 336.40	310.52	625.47	K
DKE	KED 46° 20′	KE 418.40			
X.	KEL 75° 10′	KL 460.96	691.16	365.49	L
KEL	EKL 43° 30′	EL 328.24			
XI.	LEG 75°	LG 455.80			
LEG	EGL 44° 55′.5				

The unit employed in the measurement of the base line, is the chain of 100 feet.

At the station A a true meridian line is determined, and the bearing of the side AB from this is found to be S. 75° W.

For greater facility and accuracy in plotting, the meridian through the first station A and perpendicular upon it through this point, are assumed as axes; and the co-ordinates of the vertices of the triangles in reference to these lines are computed as explained in art. 144, 145, and written in the table in the columns marked M. D. and P. D.

The meridian distances are marked with the signs + or — according as they are east or west of the first meridian; and the latitudes written in the column P.D. have the sign + or — according as they are north or south of the perpendicular upon the meridian. The direction of the triangulation is thus obviously determined.

SECONDARY TRIANGULATION.

1. The following table gives the secondary triangles for a number of stations west of the main river. The first column contains the side of the primary triangle which forms the base of the secondary. The second contains the remaining sides of the secondary triangles which we have computed. The general direction of the vertex of each triangle, as north or south &c. of the side of the primary triangle on which it depends, is indicated in the third column.

AB,	A*m* 180, B*m* 378;	*m*, S.	Signal on Island.
“	A*b* 255, B*b* 220;	*b*, S.	
“	A*e* 180, B*e* 290;	*e*, N.	
“	A*f* 279, B*f* 300;	*f*, N.	
“	A*h* 211, B*h* 336;	*h*, N.	
CD,	C*h* 197, D*h* 350;	*h*, S.	Same as preceding.
D*h*,	*hp* 280, *p*D 310;	*p*, E,	Laid off from D*h*.
DK,	D*q* 252, K*q* 133;	*q*, N.	

Detail Survey. The shore lines along the coast and west side of the main river, with both shores of the tributary, are traced by offsets from lines between the points determined by the triangulation. The work commences at the station B, and passes round through A, C, D, &c. to its termination at *q*.

A. Pt.	
134	20
115	15
95	4
55	9
0	22
d	E. fr. *b*A.
150	25
125	8
98	4
35	14
0	
b. Pt.	
160	20
130	10
75	48
35	20
0	10
c	S. fr. B*b*.
125	10
86	16
40	13
0	21
B to A	

g	N.fr. C*h*.
35	
C. Pt.	
133	15
90	00
55	9
25	14
0	
f. Pt.	
103	11
80	10
45	5
0	
e. Pt.	
180	5
150	10
130	15
90	22
55	30
25	35
0	15
A to *g*	

p. Pt.	
135	10
115	25
100	30
80	37
37	18
0	
n	S. *hp*.
168	4
150	8
118	12
75	18
35	21
0	
h. Pt.	
187	14
160	20
135	18
80	10
45	15
0	0
g to *p*.	

Breadth of river at $q = 22$

q. Pt.	
250	00
220	40
185	42
138	35
75	15
40	12
0	
D. Pt.	
310	18
275	10
250	15
235	20
162	7
118	20
65	10
25	22
0	
p to *q*.	

2. The following table exhibits the secondary triangulation for a number of points east of the main river.

GH,	G*a* 288,	H*a* 200;	*a*, E.
"	G*b* 275,	H*b* 184;	*b*, W.
FE,	F*d* 270,	E*d* 361;	*d*, W.
"	F*e* 320,	E*e* 220;	*e*, W.
AE,	A*f* 298,	E*f* 240;	*f*, N.
EK,	E*h* 373,	K*h* 275;	*h*, W.
EG,	G*p* 122,	E*p* 363;	*p*, E.
"	G*q* 275,	E*q* 225;	*q*, E.
"	G*r* 395,	E*r* 260;	*r*, E.
"	G*s* 283,	E*s* 384;	*s*, E.

Detail Survey. The shore line along the coast and the eastern bank of the river, from *a* to a point *o* on K*q*, is determined by offsets as follows.

	b. Pt.
17	184
5	130
30	40
25	0
	H. Pt.
18	200
0	110
5	34
10	0

Begin at *a*.

	c	
10	187	S. from F*d*.
34	125	
25	90	
10	30	
20	0	
	F. Pt.	
15	163	
25	80	
40	36	
	0	

b to *c*.

	e. Pt.
16	160
5	126
10	80
8	0
	d. Pt.
12	176
40	150
30	122
15	90
18	32
	0

c to *e*.

	h. Pt.
24	206
20	147
26	110
21	90
7	40
	0
	f. Pt.
10	128
0	80
0	40
18	0

e to *h*.

	210	*k* station on line *h*K
25	160	
21	88	
23	0	*h*

0	75	*o*, station on K*q*, distant 99.5 from K.
20	43	
	0	*k*; *ko* perpendicular on *h*K.

3. Island and Lake. Stations I, *m*, *p*, *q*, *r*, and *s*.

	u	
20	140	E. fr. I*m*.
13	120	
10	96	
23	70	
3	30	
10	0	
	m. Pt.	
7	248	
10	194	
17	165	
20	129	
12	104	
20	50	
25	33	
5	0	

Begin at I.

	I. Pt.
20	185
12	140
30	100
21	73
31	43
14	20
16	0

r. Pt.	
130	
116	5
75	10
60	16
35	0
0	11
q. Pt.	
180	12
152	16
130	25
115	15
86	7
24	7
0	

Begin at *p*.

p. Pt.	
166	
140	5
120	0
76	26
50	24
22	16
0	
s. Pt.	
187	
173	22
150	28
120	5
88	24
62	18
23	3
0	

The stations I and *m* determine the position of a large island on the south of the survey. The first and second columns of the table above, are the field notes by which the shore line may be traced.

The stations *p*, *q*, *r* and *s* on the east side of the river, are in the vicinity of a lake, the shore line of which may be traced from the notes in the third and fourth columns above.

4. The position of a light house south of the entrance of the river, is determined by the secondary triangle IFM, the remaining sides of which IM, FM are found to be 310, 372 respectively.

Triangles of the third order.

5. Upon the principal peaks of a mountain range north and east of the survey, signals are placed, marked P, P_1, P_2, &c., in order. To determine these points the following angles were measured at the stations K, L and G of the main triangulation.

Station K.	Station L.	Station G.
PKL, 71° 30′,	PLK, 24° 30′,	P_3GL, 17°,
P_1KL, 51° 20′,	P_1LK, 42° 15′,	P_4GL, 41° 20′.
P_2KL, 23° 30′.	P_2LK, 55° 30′,	
	P_3LG, 79°,	
	P_4LG, 61° 30′.	

6. A reef of rocks west of the side FH of the primary triangulation is determined by the following angles.

Station F.	Station H.
R FH, 79°	R HF, 32° 20′,
R_1FH, 63° 30′,	R_1HF, 42°,
R_2FH, 53°,	R_2HF, 54° 30′,
R_3FH, 37° 40′.	R_3HF, 52° 15′.

7. At the extremities of a small island, 30 in its extreme breadth, signals Q, Q_1, are placed, and the following angles observed from the stations F and *d* on the east side of the river; viz. Q*d*F 61°; Q_1*d*F 51° 20′; Q_1F*d* 56°; QF*d* 48°.

Roads, Villages, &c.

8. Commencing at a point on the lake at the distance from *q* on *qr* equal to 36, the north line of a road passes the station E at a distance of 50 N.; thence it proceeds to the river which it meets at a point on *fe* distant 55 from the secondary station *f*. At this point the river is crossed by a bridge. Commencing on H*a* at a point distant 88 from H, another road passes the station E at a point 60 E., continuing in the same direction until it meets the road above described; thence it proceeds to the secondary station *k*, near K; from which it passes along *ko* to the river.

From the junction of the two preceding roads another road leads off in a north westerly direction, passing the station L at a distanc 75 N.

At the junction of the roads is a small village. Another village is situated between *q* and *r* on the lake; and another upon the river in the vicinity of K. The point of land between the main river and its branch is the site of a large city, the principal streets of which run N. and S., E. and W.

The land through which the roads above named pass, is laid off, in general, into farms of 100 acres or more. All along the roads are the farm houses, orchards, gardens, country seats, &c.

203. The above are mere general directions. The learner will fill up the map at pleasure; sketching, by means of the proper signs, all the different objects, such as the mountain peaks, the eminences on which the primary signals are supposed to be placed, the woodlands, cultivated fields, gardens, orchards, the shores of the rivers, the villages, city, &c.

The careful filling up of the map will be found a useful exercise. We cannot go more fully into details. What has been done is sufficient to show how they may all be connected with the primary triangulation; and thus a perfect map of the entire territory be formed.

SECTION X.

GEODESIC SURVEYING.

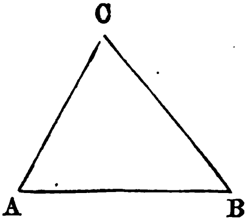
204. When a survey extends over a large portion of the earth's surface, as a whole country, or the coast of the United States for example, the work is called *Geodesic Surveying.*

A geodesic survey differs from the preceding chiefly in the greater accuracy with which all the operations must be performed; and the corrections to be applied in consequence of the curvature of the earth, which must now be taken into consideration.

Measurement of Angles.

205. The instruments employed for the measurement of the angles, especially those of the primary triangulation, should be of the largest size, and the most skillful workmanship. The horizontal circles of the theodolites employed in the U. S. Coast Survey, for the measurement of the angles of the primary triangles, are from 24 to 30 inches in diameter.

To eliminate, as far as practicable, every source of error, each angle is measured a large number of times, usually from 50 to 60, and the mean taken.



Let CAB be the angle to be measured. Place the instrument at A, and make a first or *rough* determination of the angle in the manner explained art. 112. The repetition of the measurement is then effected as follows. The telescope being still directed upon B, clamp the vernier plate to the graduated limb, and, unclamping the limb, turn the instrument back until the telescope is again directed upon C. Clamp the limb, and unclamping the vernier plate, direct the telescope again upon B. The arc passed over from the beginning, or first position of the telescope upon the side AC, will now, it is evident, be twice the angle CAB. Proceeding in like manner, the arc measured may be any number of times the angle CAB. Dividing the whole arc by the number of repetitions, we obtain the *true* or required angle. Thus, let it be supposed that there have been six repetitions of the angle; and that the mean of the verniers at the last reading is 275° 45′ 33″; the mean or true angle will be 45° 57′ 35″.5.

Reduction to the Horizon.

206. If the angles are measured in the plane of the sides of the triangle, as is usually the case in the French surveys, a correction, in general, becomes necessary in order to reduce them to a horizontal plane. This is called the *Reduction to the Horizon.* In the use of the theodolite this reduction is avoided; as, by the construction of the instrument, the angles in each triangle are all measured in the same horizontal plane.

Reduction to the Centre.

In an extended series of triangles it often becomes necessary to employ for stations steeples, towers and other similar objects. In such cases the instrument cannot be placed at the centre of the station. It is placed, therefore, as near to it as possible, and a correction applied by which the angle at the centre will be found. This is called the *Reduction to the Centre.*

Spherical Excess.

207. The triangles of which the chain is composed being on the surface of a sphere, the angles measured in each case, strictly speaking, are those of a spherical triangle; and are by consequence, greater than the sum of the angles in a plane triangle having its vertices at the same points. Whenever the sides of a triangle exceed six or eight miles this difference becomes apparent, and a correction on account of the excess must be applied, by which the angles are reduced to what they would be, if the sides were straight lines or chords of the arcs by which the vertices of the triangles are joined.

Base Line.

208. One of the most delicate operations of the survey is the measurement of the base line. Different instruments have been employed for this purpose, such as steel chains, rods of deal, glass or platinum. The greatest care is requisite in order to bring the ends of the rods or chains employed into exact contact, and so to arrange them one after the other as they are moved forward, that an exact straight line may always be preserved between the extreme points of the base to be measured. A careful record of the temperature must also be kept, in order to apply the necessary correction for changes in the length of the chains or rods, arising from the variation of temperature.

In the survey of the coast of the United States a combination of rods of different metals is employed, so adjusted that the apparatus will always maintain the same invariable length, whatever the change of temperature.

The base line is reduced to the level of the sea; that is, to its length as it would be if measured upon a horizontal plane at the level of the sea. Several base lines are also measured in different parts of the survey as a test of the accuracy of the work.

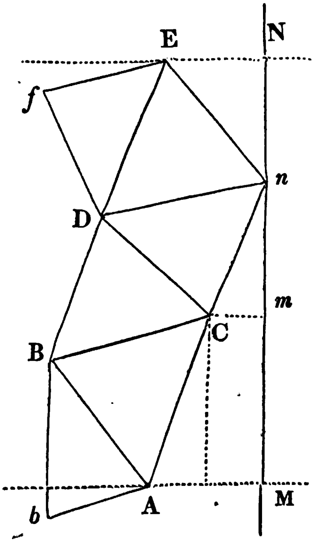
209. In addition to the operations described above, observations are made at the primary stations to determine, 1°. the azimuths or true bearings of the sides of the triangles; 2°. the latitude and longitude of each of the stations; 3°. their height above the level of the sea.

We have merely indicated the leading operations. A full description of these, with the necessary formulas would far exceed the limits of this work. A geodesic survey of the coast of the United States is now in progress, under the superintendence of Prof. A. D. Bache, conducted with consummate skill, and with all the aids which modern science has furnished for the purpose.

In this survey base lines are measured on Long Island, N. Y., on Kent Island in Chesapeake Bay, and in various other parts of the survey. As an example of the accuracy with which such surveys are conducted, it may be remarked that the Kent Island base deduced from that on Long Island through the entire series of triangles by which the two are connected, covering an extent of 200 miles, gives a difference of 20 inches only between the measured and computed base.

Arc of a Meridian.

210. By means of a geodesic survey the length of an arc of a meridian may be determined; and hence the entire circumference of the earth considered as a sphere.



Let A and E be two points at a distance from each other on the surface of the earth. We connect these points by a chain of triangles ABC, BCD, &c., and measure all the angles in each of the triangles. Let A*b* be the base line. Having measured this line, we then calculate all the sides in the triangles of the series. A second base line E*f*, at a distance from the former, will serve to verify the work.

We next determine a meridian line passing through A, for example, and from this find the azimuth or true bearing of the sides AC, C*n*, *n*E of the triangulation, and calculate the difference of latitude for each of these sides. We thus obtain the whole difference of latitude MN between the points A and E. Finally, we determine, by means of Astronomical observations, the latitude of each of the points A and E. The difference between these will be the length, in degrees, of the arc MN of a meridian. From this we shall be able to find the length of an entire meridian circle, or the circumference of the earth considered as a sphere.

Thus in the great French Survey extending from the British channel to the Mediterranean, the difference of latitude between the extreme north and south points of the survey is 12°.370303; the whole length of the arc as determined by the triangulation is 751567 English fathoms or 4509402 feet. This gives, 69.0405 miles nearly for the length of a degree; which, being multiplied by 360, gives 24854.58 miles for the circumference of the earth considered as a sphere.

Ex. 1. In Peru the latitudes of the extreme points of the survey are 0° 02′ 31″.22 S. and 3° 04′ 31″.9 N. respectively, and the measured arc comprised between them 1131060 feet. What is the length of a degree derived from this measurement?

Ans. 68.7132 miles.

Ex. 2. In the survey in Sweden the difference of latitude between the extreme parallels was found to be 1°622022, and the length of the measured arc 593220 feet. What is the length of the degree? Ans. 69.2668 miles.

By surveys of this kind the form of the earth may be determined. From measurements already made in Peru, France, England and Sweden, it is found that the degrees increase in length as we proceed from the equator to the pole. It is thus ascertained that the earth is not a sphere but is flattened toward the poles.

BOOK III. NAVIGATION.

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SECTION I.

DEFINITIONS. DESCRIPTION OF INSTRUMENTS.

211. Navigation is the art of conducting a ship from one port or place on the earth's surface to another.

The place of a ship at any time is determined by her latitude and longitude.

Latitude is distance from the equator measured on a meridian; and is either north or south.

Longitude is distance from an assumed meridian measured on the equator; and is either east or west.

Departure is the distance east or west between two places measured on a parallel of latitude.

There are two ways in which the place of a ship at sea may be determined.

1°. By keeping account of the course and distance sailed, and deducing therefrom her latitude and longitude.

2°. By calculating her latitude and longitude from observations upon the heavenly bodies.

It is the former method only which we shall consider.

The Mariner's Compass.

212. The instrument by which the course of a ship is determined is the *Mariner's Compass.*

It is the same in principle with the Surveyor's Compass. Its principal parts are 1°, A circular card, the circumference of which is divided into 32 equal parts called *points*, and these again into four equal parts called *quarter points.* 2°. A magnetized needle placed beneath the card, and so attached to it that the centre on which the needle turns is at the centre of the card, and its north and south points directly under the points marked N. and S. on the face of the card. 3°. A compass box with adjustments such that the card, when suspended on its pivot, will always maintain a horizontal position, whatever the position of the ship.

On the inside of the compass box a vertical line or mark is drawn, and the compass is so placed that a line drawn from this mark through the centre of the card, will be parallel to the keel of the ship. The needle having a free motion on its centre, the point on the card which coincides with this mark, will indicate the course of the ship.

213. The points of the compass, commencing at the north and passing round from right to left, are marked on the compass card, as follows.

1st Quad.	2d Quad.	3d Quad.	4th Quad.
N.	E.	S.	W.
N. by E.	E. by S.	S. by W.	W. by N.
N. N. E.	E. S. E.	S. S. W.	W. N. W.
N. E. by N.	S. E. by E.	S. W. by S.	N. W. by W.
N. E.	S. E.	S. W.	N. W.
N. E. by E.	S. E. by S.	S. W. by W.	N. W. by N.
E. N. E	S. S. E.	W. S. W.	N. N. W.
E. by N.	S. by E.	W. by S.	N. by W.

Beginning at the north and passing round in order, the points of the compass are read as follows; north, north by east, north-north-east, north-east by north, north-east, north-east by east, east-north-east, east by north, east, &c.

To begin at any point and name the points of the compass in order is called by seamen *boxing the compass.*

214. A point being one-eighth of a quadrant is equal to 90° ÷ 8, or 11° 15′. A quarter point is, therefore, 2° 48′ 45″. In running a course, mariners do not take into account an angle less than a quarter point.

In the following table we have the number of degrees and minutes corresponding to each point of the compass.

North.		Pts.	Deg.	South.	
N. by E.	N. by W.	1	11°15′	S. by E.	S. by W.
N. N. E.	N. N. W.	2	22°30′	S. S. E.	S. S. W.
N. E. by N.	N.W. by N.	3	33°45′	S. E. by S.	S. W. by S.
N. E.	N. W.	4	45°	S. E.	S. W.
N. E. by E.	N.W. by W.	5	56°15′	S. E. by E.	S.W. by W.
E. N. E.	W. N. W.	6	67°30′	E. S. E.	W. S. W.
E. by N.	W. by N.	7	78°45′	E. by S.	W. by S.
East.	West.	8	90°	East.	West.

The Log.

215. The rate of a ship's sailing is determined by the log. This consists of a board in the form of a sector of a circle, and loaded at the circumference so as to maintain a vertical position in the water. A line called the *log-line* is attached to it in such a manner that the broad surface, or face of the log, is kept toward the ship. Thus, by the resistance of the water, the log is kept in its place, while the line is run off from a reel, as the ship moves forward.

The length of line run off in a *half minute* gives the rate of sailing. The log line is usually divided into equal parts called *knots*. Each knot is taken equal to the 120th part of a geographical mile or degree. And since a half minute is the 120th part of an hour, the number of knots and parts of a knot run off in a half minute will express the number of miles and parts of a mile which the ship makes an hour, at the same rate of sailing.

The process of determining the rate of sailing by the log is called by seamen, heaving the log.

Rhumb Line.

216. The line of rhumbs on Gunter's scale is constructed in a manner similar to the line of chords. The quadrant is divided into eight equal parts corresponding to the points of the compass, and these again into four equal parts for the quarter points. The chords of these several divisions are then transferred to the same line, and are duly marked and numbered, as in the case of the line of chords.

The use of the line of rhumbs is to lay down the course of a ship in points and quarter points of the compass. The radius of the line of rhumbs will be the chord of 60°.

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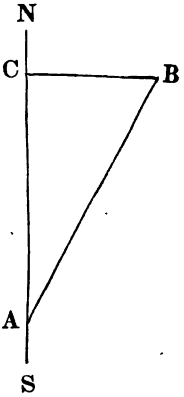
SECTION II.

PLANE SAILING.

217. For short distances the earth's surface may, without sensible error, be considered as a plane, and the path of a ship, her course, difference of latitude and departure may be regarded as parts of a right-angled plane triangle.

Ex. A ship in latitude 37° 10′ N. sailed N. N. E. 80 miles. What is the departure made, and the latitude of the ship?

1°. *By construction.* We draw an indefinite line NS, to represent the meridian from which the ship departs. Assuming the point A as the place of departure, we lay off by aid of the line of rhumbs, the angle NAB equal to the course, which in this case is 2 points. Through this we draw AB = 80, the distance sailed, and from B let fall BC perpendicular upon NS; AC will be the difference of latitude and BC the departure, which may now be measured upon the same scale of equal parts from which the distance AB was laid down.



2°. *By computation.* The same figure being supposed we have

As rad.		10.000000
: Distance 80		1.903090
: : Sin Course 22°. 30′		9.582840
: Departure	30.61	11.485930

Thus the departure is 30.61 miles. In like manner we find the difference of latitude = 73.91 miles. Dividing this by 60, since sixty geographical miles make a degree, we have 1° 13′ 55″ for the difference of latitude in degrees. This being added to 37° 10′, the latitude of the point of departure, gives 38° 23′ 55″ for the latitude of the ship.

218. The right angle being always known, of the four things, course, distance, difference of latitude, and departure, the others may be found when any two of them are given.

I. Given the course and distance.

Ex. A ship sails from latitude 47° 30′ N. upon a course S. W. by S. 98 miles. What latitude is she in, and what departure has she made? Ans. Lat. 46° 8′ N. Dep. 54.45 miles.

II. Given the course and difference of latitude.

Ex. A ship in latitude 23° 7′ S. sails E. N. E. until she comes to the latitude of 22° 10′ S. What is the distance sailed and departure? Ans. Dist. 149 miles. Dep. 137.6 miles.

III. Given the course and departure.

Ex. A ship in latitude 16° 35′ S. sailed on a course N. E. ½ N. until she found her departure to be 342 miles. What was the distance sailed and the latitude at which she arrived?

Ans. Dist. 540 miles. Lat. 9° 38′ S.

IV. Given the distance and difference of latitude.

Ex. A ship from latitude 1° 4′ S. sails between the N. and W., that is, north-westerly, 162 miles, and then finds her latitude by observation to be 52′ N. Required her course and departure.

Ans. Course N. 44° 15′ W. Dep. 113 miles.

V. Given the distance and departure.

Ex. A ship sails on a southerly course 118 miles, and makes 83 miles westing. Required her course and difference of latitude.

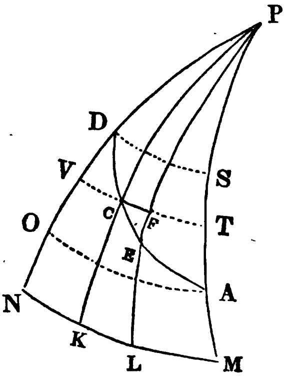
Ans. Course S.44° 42′ W. Diff. Lat. 83. 8 miles.

VI. Given the difference of latitude and departure.

Ex. A ship in latitude 30° 5′ north sails south-westerly until she has made a difference of latitude of 13.3 miles, and her departure is 19.9 miles. What is her course and distance sailed?

Ans. Course S. W. by W. Dist. 24 miles.

219. The process above is true not only for short distances in which the surface of the earth may be considered as a plane, but for all distances when the course is unchanged.



To demonstrate this let AD represent the path described by the ship or the distance sailed, P the pole of the earth, PN and PM the extreme meridians passing through A and D, NM a portion of the equator, and DAS the angle of the course.

Since the ship makes the same angle with each meridian she passes while sailing constantly on the same course, her path, it is evident, will be a curve line. This curve, moreover, is not a portion of a great circle; but is a peculiar kind of curve, called a *rhumb-line* or *loxodromic curve.*

Suppose next AD divided into any number of equal parts so small that any one of them, CE for example, may be regarded as a straight line. Through C and E draw the meridians PK, PL, and draw CF parallel to NM. Through A and D draw also the parallels AO and DS. The triangle CEF will be right angled at F, and may be regarded as a plane triangle. In like manner, let triangles be conceived to be constructed on all the parts of AD equal each to CE. The sum of the parts CE, or the sum of the hypothenuses of the small triangles will, it is evident, be equal to AD or the distance sailed; and the sum of the sides EF will be equal to AS the difference of latitude; and we shall have the proportion,

sum of the sides CE : sum of the sides EF : : CE : EF

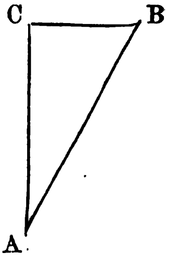
or AD : AS : : CE : EF. (1)

Let next a right-angled triangle be constructed in which the hypothenuse AB is taken equal to AD the distance sailed by the ship, and the angle BAC equal to the course DAS. This triangle will be similar to CEF, and we have

AB : AC : CE : EF ; (2)

whence comparing (1) and (2)

AD : AS : : AB : AC ; (3)



in which if AB is equal to AD, AC will be equal to AS.

Notwithstanding, therefore, the path described by the ship is a curve line, if we construct a right-angled plane triangle, in which the hypothenuse is equal to the distance sailed and one of the acute angles equal to the course, *the side adjacent to this angle will be equal to the difference of latitude whatever the distance sailed.*

By a process altogether similar it may be shown that the side BC or that opposite the course, will be equal to the sum of the departures CF or the whole departure.

We thus see that all the cases of plane sailing, in which the distance, course, difference of latitude and departure are the only things considered, may be solved by aid of a right angled plane triangle.

220. Thus far we have considered the place of a ship determined by the difference of latitude and departure. To know, however, her actual position we must be able to convert the departure into difference of longitude, in order that her longitude may also be found.

Suppose that a ship sails from D to S (fig. art. 219) on the parallel of latitude DS. In this case the departure DS will be equal to the distance sailed; and to determine the actual change of place of the ship, east or west, we must find the difference of longitude, or length of the arc NM on the equator, corresponding to the departure DS.

In order to this it will be recollected that similar arcs are as their radii; whence DS is to NM, as the radius with which DS is described to the radius with which NM is described. Regarding the latter as R, or the sine of 90°, the radius of the tables, the former will be the sine of DP, or the cosine of ND; whence

$$\cos ND : R :: DS : NM,$$

or, *The cosine of the latitude is to radius as the departure is to the difference of longitude.*

221. When a ship sails exactly east or west on the same parallel, the only change made in her position is in the longitude which may be correctly determined by means of the above proportion. This case is usually called *parallel sailing.*

Ex. 1. A ship sailed from Boston, in longitude 71° 4′ 9″ W., 1000 miles exactly east; what is the longitude at which she has arrived? Ans. 48° 32′ W.

Ex. 2. What is the departure in the latitude of 45° corresponding to one degree of longitude? Ans. 42.43 miles.

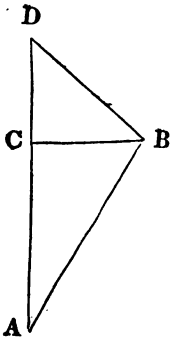
SECTION III.

MIDDLE LATITUDE SAILING.

222. A ship ordinarily changes both her latitude and longitude at the same time, and the question returns in this case to find the difference of longitude. Thus suppose AD (figures art. 219) the distance sailed on an oblique course DAS. We have seen that AB being taken equal to the distance AD, and the angle BAC = DAS, CB will be equal to the sum of the departures CF, or to the whole departure. But on what parallel shall this departure be reckoned? It is evident that it cannot be reckoned on AO, the parallel passing through A the point from which the ship departs; for, in consequence of the convergency of the meridians, the sum of the sides CF is manifestly less than AO; nor can it be reckoned on the parallel DS, the parallel passing through D the point at which the ship has arrived; for the sum of the sides CF, for the same reason as before, is greater than DS. On the whole it will be sufficiently near, in general, to reckon the sum of the sides CF, or the departure, upon the middle parallel VT, or that which passes half way between AO and DS. It is this in which the principle of middle latitude sailing consists.

Upon CB in the triangle CAB construct the triangle CBD in which the angle CBD is taken equal to the middle latitude; then BD will be equal to the difference of longitude, since we have

sin CDB or cos CBD : R : : CB : BD.



223. The two triangles CAB, CBD constitute together a single triangle ABD. From the three we obtain the following proportions.

I. From the triangle CBD

sin CDB, or cos CBD : CB : : R : BD ;

or in words, *The cosine of the middle latitude is to the departure, as radius to the difference of longitude.*

II. From the triangle ABD

sin ADB : AB : : sin BAD : BD ;

or, *The cosine of the middle latitude is to the distance, as the sine of the course to the difference of longitude.*

III. From the triangle BAC

R : tan BAC : : AC : CB ;

comparing this with the first proportion, and observing that the extremes in this last are the means in the first, we obtain

AC : BD . : cos CBD : tan BAC,

or, *The difference of latitude is to the difference of longitude, as the cosine of the middle latitude to the tangent of the course.*

224. The three proportions above comprise all that is necessary for the solution of the different cases in middle latitude sailing. We subjoin a few examples, observing that in these, as well as in the remaining examples of this work, longitude is reckoned from the meridian of Greenwich.

I. Given the latitude and longitude of the place of departure, and the course and distance.

Ex. A ship in latitude 47° 23′ N., and longitude 10° 17′ W. sails 126 miles on a course S. W. by W. What is the latitude and longitude of the place at which she arrives?

We first find, as in plane sailing, the difference of latitude = 70 miles, or 1° 10′. This being subtracted from 47° 23′, the latitude left, gives 46° 13′ for the latitude in.

Adding next half of 1° 10′ or 35′ to the less latitude 46° 13′, we obtain the middle latitude 46° 48′. Then by the second proportion we obtain the difference of longitude equal to 153 miles or 2° 33′. And as the direction in which the ship sailed is from the first meridian, adding this to 10° 17′, the longitude of the place of departure, we obtain 12° 50′ W. for the longitude of the place at which she arrives.

II. Given both the latitudes and longitudes.

Ex. A ship in latitude 29° 40′ S. and longitude 53° W. sails on a northwest course until by observation her latitude is found to be 26° 13′ S., and longitude 55° 38′ 36″ W. What is her course and the distance sailed?

Ans. Course N. 34° 6′ W. Dist. 250 miles.

III. Given the latitude and longitude of the place left, the latitude in and the departure.

Ex. A ship in latitude 38° 10′ N. and longitude 56° W. sails on a south-easterly course until she is found by observation to be in latitude 37° 11′ N., having made 54 miles departure. What is the longitude in and the course on which she has sailed?

Ans. Long. 54° 51′ 48″. Course S. 42° 28′ E.

IV. Given the latitude and longitude left, the latitude in and the course.

Ex. A ship in latitude 42° 30′ N., and longitude 41° 45′ W. sails S. E. by S. until she finds her latitude to be 38° 21′ N. Required the distance sailed and longitude in.

Ans. Dist. 299.5 miles. Long. 38° 6′ 30″ W.

V. Given the longitude left, both latitudes and the distance.

Ex. A ship in latitude 23° 11′ N. and longitude 53° 38′ W. sails on a north-westerly course 230 miles, when she finds her latitude by observation 25° 43′ N. Required her course and longitude in. Ans. Course N. 43° 38′ W. Long. 56° 47′ 36″.

VI. Given the latitude and longitude left, the course and departure.

Ex. From a port in latitude 53° 7′ N. and longitude 63° 15′ W., a ship sails E. S. E. until her departure is 112 miles. What is the latitude and longitude in?

Ans. Lat. 52° 21′ N. Long. 60° 10′ 6″ W.

VII. Given the latitude and longitude left, the distance and departure.

Ex. A ship from Sandy Hook in latitude 40° 28′ N. and longitude 74° 1′ W. sails south-easterly 128 miles, having made 103 miles departure. Required the place of the ship.

Ans. Lat. 39° 12′ N. Lon. 71° 46′ 54″ W.

The method of middle latitude sailing gives the difference of longitude with sufficient accuracy for moderate distances near the equator, where the convergency of the meridians is small. It becomes increasingly more inaccurate as we approach the poles. A correction, however, may be applied which will render the process correct for all latitudes. Tables for this purpose have been prepared by Workman.

SECTION IV.

TRAVERSE SAILING.

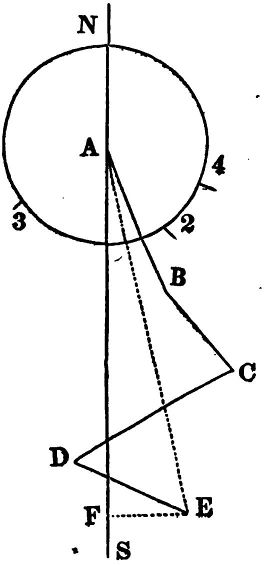
225. Thus far we have supposed the ship to sail on a single course. But ordinarily, in going from one place to another, a ship sails on several different courses. And the question is to find the single course and distance equivalent to the several courses and distances which the ship has run.

The irregular track made by the ship is called a *Traverse ;* and the finding the single course and distance equivalent to it, is called compounding or *working the Traverse.*

To find the whole difference of latitude made by the ship we must find, it is obvious, the several differences of latitude made upon the courses run separately ; then subtracting the sum of the differences corresponding to the southerly courses from the sum of the differences corresponding to the northerly courses, or the converse as the case may require, we shall have the difference of latitude on the whole made by the ship ; or, as it is sometimes expressed, the difference of latitude made good. In like manner we find the departure on the whole made by the ship. With the difference of latitude and departure thus found we construct a right angled triangle, the hypothenuse of which will be the equivalent distance, and the acute angle adjacent to the difference of latitude the equivalent course sought.

Ex. 1. A ship from a certain port in latitude 42° N. sails on the following successive tracks; 1, S. S. E. 80 miles; 2, S. E. 55 miles; 3, S. W. by W. 115 miles; 4, E. S. E. 62 miles. Find the course and distance for the whole traverse.

1°. *By construction.* With the chord of 60° from the line of chords, we draw a circle to represent the horizon, and which we call the horizon circle. Through the centre A of this circle we draw indefinitely a line NS, to represent the meridian passing through A the point of departure. We next, by aid of the rhumb line, lay off on the horizon circle the several courses, numbering them 1, 2, 3, &c., in order. In the present case, the courses being all southerly are laid off from the south point S, those which are easterly to the right, and those which are westerly to the left. This being done, from A in the direction A1 we draw AB equal 80 miles the distance sailed on the first course; next from B in a direction parallel to A2 we draw BC equal 55 miles the distance sailed on the second course, and so on to DE the distance sailed on the last course. Then E will be the place of the ship at the end of the traverse. Letting fall next from E the perpendicular EF, AF will be the difference of latitude and FE the departure made



good. Joining AE, AE will be the distance and FAE the course equivalent to the traverse.

2°. *By computation.* It is convenient to arrange the data and calculations in a tabular form analogous to what we have done in surveying. Thus, for the present example, we have the following table which sufficiently explains itself.

No.	Course.	Dist.	N.	S.	E.	W.
1	S. 22° 30′ E.	80 mls.		73.9	30.6	
2	S. 45° E.	55		38.9	38.9	
3	S. 56° 15′ W.	115		63.9		95.6
4	S. 67° 30′ E	62		23.7	57.3	
Sum of columns.				200.4	126.8	95.6

Subtracting the westing from the easting we obtain 31.2 miles for the departure made good. From this with 200.4 miles the difference of latitude made good, we find, by plane sailing, the course S. 8° 51′ E; and the distance 202.8 miles.

226. The course and distance being thus found, the difference of longitude may next be found by middle latitude sailing.

Ex. 2. A ship sails on the following successive courses; 1, S. S. E. 31 miles; 2, S. E. by E. 27 miles; 3, S. W. 42 miles; 4, W. by N. 34 miles; 5, S. by E. 50 miles. What is the single course and distance equivalent ?

Ans. Course S. 9° 45′ 33″ W. Dist. 117.4 miles.

Ex. 3. A ship is bound to a port distant 100 miles from the place of departure and bearing W. by S. She has sailed on the following courses; 1, S. 10 miles; 2, W. S. W. 25 miles; 3, S.W. 30 miles; 4, W. 20 miles. What is now her direct distance from the place of her destination, and upon what single course must she sail in order to reach it ?

Ans. Dist. 40 miles. Course N. 57° 47′ W.

Ex. 4. A ship from latitude 43° 25′ N.; longitude 63° W. sails on the following courses; 1, S. W. by S. 63 miles; 2, S. S. W. ½

W. 45 miles; 3, S. by E. 54 miles; 4, S. W. by W. 74 miles. Required the place of the ship.

Ans. Lat. 40° 19′ N. Long. 65° 23′ 48″ W.

Ex. 5. A ship in 17° 12′ N. latitude bound to a port in 18° 40′ N. latitude and 220 miles to the westward, sails on the following courses; 1, N. W. by W. 73 miles; 2, W. N. W. 40 miles; 3, S. S. W. 18 miles. What is now her distance from the place of her destination, and on what single course must she sail in order to reach it? Ans. Dist. 126 miles. Course N. 67° W.

Ex. 6. A ship at noon on a certain day was in latitude 10° N. and longitude 41° W. She then sailed on the following courses; 1, N. by E. 25 miles; 2, N. N. E. 32 miles; 3, N. W. 28 miles; 4, N. W. by N. 35 miles. Required the distance sailed on a direct course, and the place of the ship.

Ans. Dist. 105.4 miles. Lat. 11° 43′ N. Long. 41° 22′ 30″ W.

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SECTION V.

MERCATOR'S SAILING.

227. The practical problem in Navigation is to determine accurately, by means of a plane triangle, the difference of latitude and longitude, when the course and distance are known.

The problem has been solved, art. 219. for the difference of latitude, but not for the difference of longitude, which, art. 222, is found only by approximation. And the question returns to find the difference of longitude, by aid of the right angled triangle, in a manner that will be accurate for all latitudes and all courses on which a ship may sail.

Returning to the figure, and putting l for the latitude of CF, we have

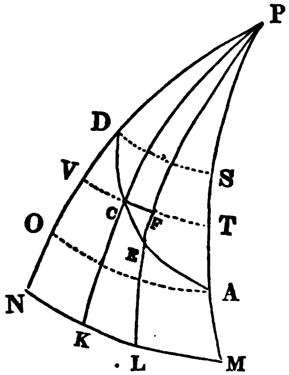
$$CF : KL :: \cos l : R,$$

or $$CF : KL :: R : \sec l;$$

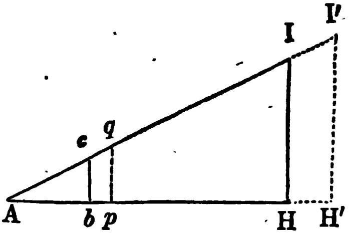
whence putting d for the departure CF, and D for KL the corresponding difference of longitude, we have, radius being unity,

$$D = d \sec l;$$

or, *The difference of longitude is equal to the departure multiplied by the secant of the latitude.*



Let us now construct a right angled triangle AHI in which AI is taken equal to the distance AD, and the angle HAI equal to the course DAS; then, as before, AH will be equal to the sum of the sides EF of the elementary triangles CEF, or the difference of latitude AS; and HI to the sum of the sides CF of these triangles or the whole departure.



Let A$b$$c$ be one of these elementary triangles equal to CEF; and let the side Ab be extended until, l being the latitude of bc, we have $$Ap = Ab \sec l.$$

Draw pq parallel to bc; then from the similarity of the triangles Abc, Apq, we shall have

$$pq = bc \sec l.$$

that is, pq will be equal to the difference of longitude corresponding to the departure bc.

Suppose next the sides in all the elementary triangles CEF extended in the same manner as those of Abc; and let the side AH of the triangle AHI be extended to H′, so that AH′ will be equal to the sum of the extended sides Ap of the elementary triangles CEF. Then if from H′ we draw HI′ parallel to HI and meeting AI produced in I′, H′I′ will, it is evident, be equal to the sum of the extended sides pq of the elementary triangles CEF, or to the true difference of longitude corresponding to the sum of the sides CF of these triangles, or the whole departure HI.

228. The extended difference of latitude AH′ may be found by calculation. It is equal, as we have seen, radius being unity, to the sum of the sides Ap of the elementary triangle CEF, or the sum of the parts into which AS, the difference of latitude, is divided multiplied each by the secant of its latitude.

To facilitate finding the extended difference of latitude corresponding to any proper difference of latitude, tables have been prepared, the construction of which may be explained as follows.

Commencing at the equator, in the formula $\mathrm{A}p = ab \sec l$ let $ab = 1'$; then $\mathrm{A}p = \sec 1'$; that is, the extended difference of latitude, or which is the same thing, the extended meridian corresponding to the first minute of latitude is the natural secant of one minute; that corresponding to the second minute is the natural secant of two minutes; and so on for each minute on the meridian from the equator to the poles.

The extended meridian corresponding therefore, to 1, 2, 3, &c., minutes of latitude will be

For 1 minute $\sec 1' = 1.0000000$
2 minutes $\sec 1' + \sec 2' = 2.0000002$
3 minutes $\sec 1' + \sec 2' + \sec 3' = 3.0000006$
4 minutes $\sec 1' + \sec 2' + \sec 3' + \sec 4' = 1.0000013$

The values of the extended meridian corresponding to 1, 2, 3, 4, &c. minutes are called *meridional* parts. They are easily found as above, and are usually set down in tables for every minute and degree of the quadrant from 0 to about 80°.

To find from the tables the meridional difference of latitude corresponding to any proper difference of latitude, if the extreme latitudes are both north or both south we subtract the meridional parts corresponding to the less latitude from those corresponding to the greater; if the extreme latitudes are one north and the other south we add together the meridional parts corresponding to each; the result will be the meridional difference of latitude sought.

229. Having thus seen the manner in which the meridional difference of latitude corresponding to any proper difference of latitude may be found, we will now show the use which may be made of it in accomplishing the object proposed.

In the triangles AHI, AH′I′ (fig. art. 227), we have

$$AH : HI :: AH' : H'I',$$

or, *The proper difference of latitude is to the departure, as the meridional difference of latitude is to the difference of longitude.*

By means of the meridional difference of latitude, we can, it is evident, by proportions in plane triangles, find the difference of longitude, when the course and distance are given. Thus the problem proposed is completely solved.

Returning again to the figure, we have, it is evident,

$$R : \text{tang } HAI :: AH' : H'I',$$

or, *Radius is to the tangent of the course as the meridional difference of latitude is to the difference of longitude.*

230. These are the fundamental proportions in Mercator's sailing. From them others may be easily derived, adapted to the different cases which may occur. We subjoin some examples.

Ex. 1. A ship in latitude 47° 23′ N. and longitude 10° 17′ W. sails 126 miles on a course S. W. by W. What is the longitude at which she has arrived?

This example has already been solved, case I. art. 193. The difference of latitude there found, 1° 10′ subtracted from 47° 23′ gives 46° 13′ N. for the latitude in.

Meridional parts 47° 23′ = 3236.6
" " 46° 13′ = 3134.3

Merid. diff. lat. = 102.3

R	10.000000
: tang. 56°15′	10.175107
: : Merid. diff. lat. 102.3	2.009876
: Diff. long. 153.1 = 2°33′6″	2.184983

Ans. Long. in 12° 50′ 6″.

Ex. 2. A ship in latitude 33° 20′ N., and longitude 27° 10′ W. sails south-easterly, until by observation her latitude is found to be 25° 30′ N; having made 231 miles departure. What is the longitude in and the course on which she has sailed?

Merid parts 33° 20′ = 2123.4; 25° 30′ = 1583.2

Diff. lat. 470	2.672098
: Depart. 231	2.363612
: : Merid. diff. lat. 540.2	2.732555
: Diff. long. 265.5	2.424069
Merid. diff. lat.	2.732555
: Diff. long.	2.424069
: : R	10.000000
Tang. Course 26° 10′ 25″	9.691514

Ans. Long. 22° 44′ 30″ W. Course S. 26° 10′ 25″ E.

Ex. 3. A ship in sight of the Peak of Teneriffe, bearing N. N. E. finds her latitude by observation 27° 5′ N. Required the longitude in.

Teneriffe latitude 28° 25′ N. Long. 16° 25′ W.

Merid. parts 28° 25′ = 1779.5
" " 27° 5′ = 1689.1

Ans. 17° 2′ 30″ W.

Ex. 4. A ship, after taking her departure from Cape Clear, sails S. S. W., and finds by observation that her longitude is 10° 54′

W. What distance has she sailed, and what is her present latitude?

Cape Clear, lat. 51° 12′ N. Long. 9° 45′ W.

Tang. 22° 30′ : R : : Diff. long. 69° : Merid. diff. lat. = 166.6

Meridional parts 51° 12′ = 3587.9

Subtract Merid. diff. lat. = 166.6

Latitude in, Merid parts = 3421.3

Seeking in the table of meridional parts for the degrees and parts of a degree corresponding to 3421.3, we find 49° 25′ 31″.

Ans. Latitude in 49° 25′ 31″. Dist. 115.2 miles.

Ex. 5. It is required to find the bearing and distance from Land's End to Bermudas, the latitude of the former being 50° 06′ N. and the longitude 6° W; and the latitude of the latter place 31° 20′ N. and the longitude 64° 48′ W.

Meridional parts 50° 6′ = 3483.8

" " 31° 20′ = 1981.4

Ans. Bearing 66° 55′ W. Dist. 2872 miles.

MISCELLANEOUS EXAMPLES.

1. A ship sailing on a N.W. course, at the rate of 5 knots an hour, observes a headland bearing E., and four hours afterward the same headland bore S. 78° E. What was its distance from the ship at time last observed? Ans. 68 miles.

2. Two ships sail from the same port. The first sails N. E. ½ E. 16 miles; the second sails easterly 20 miles, and then finds that the first bears N. N. W. Required the course of the second ship and the distance between the two ships.

Ans. Course S. 72° 27′ E. Dist. 17.5 miles.

3. From a ship under sail, an island is observed to bear N. 22½° E., and after proceeding N. 67½° W. 20 miles, the bearing of the same island is found to be N. 56¼° E. The distance from each place of observation is required.

Ans. 29.93, and 36 miles respectively.

BOOK IV.

TRIGONOMETRICAL ANALYSIS.

231. In the investigation of rules for the solution of a plane triangle a difficulty, as we have seen, arises on account of the necessity of taking into consideration the magnitude of the angles. This difficulty has been obviated by employing as auxiliaries certain lines, which bear a determinate relation to the angles, and the values of which, having been computed, are arranged in tables. These lines, called sines, cosines, tangents, &c., are, in general, termed *trigonometrical lines.*

Beside the simple relations between the trigonometrical lines already investigated, and which are sufficient for the solution of a plane triangle, others exist, giving rise to a great variety of formulas of the greatest utility in the higher operations of Surveying, in the investigations of Physical Astronomy, and the Physical Sciences generally.

These formulas are most readily derived by the aid of algebra; and that part of trigonometry which relates to their determination, in the use of this instrument, is called *Analytical Plane Trigonometry.*

232. In the solution of plane triangles the arcs which come under consideration never exceed a semi-circumference; and we have occasion to consider only the numerical values of the sines, cosines, tangents, &c. But in the more extended applications of the trigo-

nometrical lines, the arcs which occur assume every possible magnitude; and we shall have occasion to consider not only the changes in the *numerical values*, but also those which take place in the *position* of these lines, as the arcs upon which they depend and of which they are regarded as functions, vary in magnitude.

233. To understand clearly the nature of these changes let us attend to the variations, which take place in the numerical values, and in the positions of the sines, cosines, tangents, &c. of an arc, while it increases from 0 to an entire circumference.

In the circle ABDE (fig. 34, Plate III.) let the diameters AD, BE, be drawn perpendicular to each other; the circle will then be divided into quadrants. The angle ACB is called the *first* quadrant, the angle BCD the *second* quadrant, the angle DCE the *third*, and the angle ECA the *fourth* quadrant.

To trace the changes in question, let the radius CF, at first coinciding with CA, be supposed to revolve about the point C, as upon a pivot, and departing from the position CA, so that the arc AF shall have successively all magnitudes whatever from 0 to an entire circumference, and the angle ACF shall increase from 0 to four right angles. In the first quadrant, when the radius CF coincides with CA, so that the arc AF is 0, the sine GF, it is evident, is also 0, while the cosine CG is equal to radius. When the radius CF moves off from CA, the sine GF increases as the point F advances toward B, until, when the point F has arrived at B, the sine GF coincides with CB, and becomes equal to radius. At the point B the arc AF is equal to a quadrant, and the angle ACF to a right angle. Under the same circumstances the cosine CG constantly decreases, until when the point F coincides with B, the cosine becomes equal to 0. In the second quadrant as the point F′ moves on from B to D, the sine G′F′ decreases, and the cosine CG′ increases, until, when the point F′ coincides with D, and the arc AF′ becomes equal to a semi-circumference, the sine G′F′ is 0, and the cosine CG′ is equal to radius.

It may be remarked in passing, that the lines G′F′, CG′ are respectively the sine and cosine of the arc DF′, and also of its supplement AF′ ; whence *the absolute magnitude of the sine and cosine of an obtuse angle is the same with that of its supplement.*

In the third quadrant, as the point F″ moves on from D to E, the sine G″F″ increases, and the cosine CG″ decreases, until at the point E, where the arc AF″ becomes equal to three fourths of a circumference, the sine is equal to radius and the cosine to 0. From E to A the sine G‴F‴ decreases and the cosine CG‴ increases, until at A, when the whole revolution has been completed, and the arc AF‴ becomes equal to an entire circumference, the sine is reduced to 0, and the cosine becomes equal to radius.

234. Observing in like manner the changes, which occur in the magnitude of the tangents through the course, which has been described, we find, that from A to B the tangents increase continually, as the arc AF increases; at the point B where the arc AF becomes equal to a quadrant, the secant CM coinciding with CB is parallel to the tangent AM, and therefore no longer meets it; so that the arc AB has not properly speaking a trigonometrical tangent. We say, indeed, that the tangent of AB or 90° is *infinite* ; but by this expression we mean, that if the difference between an arc and 90° is indefinitely small, the tangent of this arc will be indefinitely great; that is, greater than any assignable quantity. From B to D the tangents decrease, until at D the tangent is 0; from this point they increase, until at E the tangent is again infinite; from E they decrease, and the tangent becomes 0 at A.

Tracing also the changes which occur in the secants, in the first quadrant the secant is equal to radius at A, and going on to increase through this quadrant it becomes infinite at B; it then decreases through the second quadrant until it becomes equal to radius at D. From this point it again increases through the third quadrant until it becomes infinite at E; from which through the fourth quadrant it decreases and becomes again equal to radius at A.

By a similar course it will be seen that the cotangent is equal to 0 at B and E, and is infinite at A and D. The cosecant is equal to radius at B and E, and infinite at A and D.

235. Let us now observe the changes in the position of the trigonometrical lines in the course which has been pursued.

In the first and second quadrants the sines, it is evident, all lie *above* the diameter AD from which they are measured; while in the third and fourth quadrants they fall *below* this line. Regarding the former as *positive*, the latter, according to the usual notation of algebra, must be regarded as *negative*.

The cosines in the first and fourth quadrants are all situated on the *right* of the centre C to which they are referred; and in the second and third they are situated to the *left* of this point. Regarding the cosines in the first of these positions as *positive*, in the second they will be *negative*. In respect to the tangents it will be seen, that from A to B the radius CF produced meets the indefinite tangent at A *above* the diameter AD; and from B to D it meets the indefinite tangent *below* this line. From D to E it again passes above the diameter AD, and from E to A it falls below.

236. The algebraic signs having been determined for the sines and cosines, the signs for the tangents are necessarily determined by these. Indeed, since the tangent is equal to the sine divided by the cosine, it follows that if the sine and cosine have both the same sign, that of the tangent will be *positive*; but if the sine and cosine have different signs, that of the tangent will be *negative*. Thus in the first and third quadrants the tangents will be positive; in the second and fourth they will be negative.

The change in the algebraic sign in passing from one quadrant to another corresponds, it is evident, to the change of position in the tangents.

In a manner altogether similar, we determine the algebraic signs for the cotangents, secants and cosecants.

237. Designating a quadrant by Q, the signs for the trgonometrical lines will be, in the different quadrants as follows.

	First Q.	*Second Q.*	*Third Q.*	*Fourth Q.*
Sine	+	+	—	—
Cosine	+	—	—	+
Tangent	+	—	+	—
Cotangent	+	—	+	—
Secant	+	—	—	+
Cosecant	+	+	—	—

238. We have traced the changes which occur with respect to the sines, cosines, &c. from the departure of the point F from A until its return to this point again. We may now suppose that at A it commences a second revolution, and regarding as one arc the whole course passed over by the point F from the commencement of its motion, we shall have arcs, that exceed a circumference, and which have the same sines, cosines, tangents &c. as those described in the first revolution.

239. In addition to the triginometrical lines above described another is frequently found of use, called the *versed sine.* It is that part of the radius comprehended between the arc and the foot of the sine; and is equal to the radius minus the cosine. Thus, ver-sin AF = R — cos AF. In consequence of the relations of these several lines to the circle, they are called also *circular functions.*

240. We now proceed to find the formulas proposed. In order that algebra may be applied to this object, certain fundamental principles must first be demonstrated by the processes of Geometry. We shall require only the known property of the right angled triangle, viz. the square of the hypothenuse equal to the sum of the squares of the two sides, and the formulas for the sine and cosine of the sum and difference of two arcs, demonstrated art. 32, 33.

Let a and b denote any two arcs, we shall have, then, for the algebraic expression of these principles, radius being 1,

$$\sin^2 a + \cos^2 a = 1 \qquad (1)$$

$$\sin (a + b) = \sin a \cos b + \sin b \cos a \qquad (2)$$

$$\sin (a - b) = \sin a \cos b - \sin b \cos a \qquad (3)$$

$$\cos (a + b) = \cos a \cos b - \sin a \sin b \qquad (4)$$

$$\cos (a - b) = \cos a \cos b + \sin a \sin b \qquad (5)$$

The notation $\sin^2 a$, $\cos^2 a$ in the first formula denotes the square of the sine of a, the square of the cosine a, &c.

The four last formulas, though derived originally from an arc comprised between 0 and a quadrant, become general when we take into consideration the position of the trigonometrical lines, or the algebraic signs which have been attributed to them.

241. Let us now examine some particular cases derived from these formulas.

1. Let $a =$ a quadrant or $\frac{1}{2}\pi$, b being less than a quadrant. Then since $\sin \frac{1}{2}\pi = 1$; and $\cos \frac{1}{2}\pi = 0$, we have

$$\sin (\tfrac{1}{2}\pi + b) = \sin \tfrac{1}{2}\pi \cos b + \sin b \cos \tfrac{1}{2}\pi = \cos b.$$

2. In like manner we obtain $\cos (\frac{1}{2}\pi + b) = -\sin b$.

To verify this last from the figure, let $b = \mathrm{BF}'$ (fig. 34). Then, since the arc $\mathrm{AB} = \frac{1}{2}\pi$, the arc $\mathrm{AF}' = \frac{1}{2}\pi + b$. But the cosine of AF′ is CG′, which is manifestly equal to the sine of BF′ or b. The cosine is negative, as it should be, since it falls to the left of the point C.

3. Let $a = \pi$; then $\sin (\pi + b) = -\sin b$, and $\cos (\pi + b) = -\cos b$.

3. Let $a = \frac{3}{2}\pi$; then $\sin (\frac{3}{2}\pi + b) = -\cos b$, and $\cos (\frac{3}{2}\pi + b) = \sin b$.

The absolute values of these last may be verified in the same manner as the preceding.

Formulas for the sines and cosines of multiple arcs.

242. Putting in the expressions for $\sin (a+b)$, $\cos (a+b)$, nos (2) and (4) of the fundamental formulas, $b=a, 2a, 3a$, &c., successively, we obtain

$$\begin{aligned}
\sin 2\,a &= 2 \sin a \cos a \\
\sin 3\,a &= \sin a \cos 2\,a + \sin 2\,a \cos a \\
\sin 4\,a &= \sin a \cos 3\,a + \sin 3\,a \cos a \\
\sin 5\,a &= \text{\&c.} \\
\cos 2\,a &= \cos^2 a - \sin^2 a \\
\cos 3\,a &= \cos a \cos 2\,a - \sin a \sin 2\,a \\
\cos 4\,a &= \cos a \cos 3\,a - \sin a \sin 3\,a \\
\cos 5\,a &= \text{\&c.}
\end{aligned}$$

Substituting for multiple arcs their values in terms of the simple arc, as $2 \sin a \cos a$ for $\sin 2a$ for example, we obtain

$$\begin{aligned}
\sin 3\,a &= 3 \sin a \cos^2 a - \sin^3 a \\
\mathrm{sih}\, 4\,a &= 4 \sin a \cos^3 a - 4 \sin^3 a \cos a \\
\sin 5\,a &= \text{\&c.} \\
\cos 3\,a &= \cos^3 a - 3 \sin^2 a \cos a \\
\cos 4\,a &= \cos^4 a - 6 \sin^2 a \cos^2 a + \sin^4 a \\
\cos 5\,a &= \text{\&c.}
\end{aligned}$$

By means of formulas thus derived we may find, it is evident, the sine and cosine of any multiple of an arc, when the sine and cosine of the arc are given.

Formulas for the sines and cosines of half a given arc.

243. In the expression for $\cos 2\,a$ art. 242, making $a=\frac{1}{2}\,a$,

we have $\cos^2 \frac{1}{2}\,a - \sin^2 \frac{1}{2}\,a = \cos a$ (1)

but art. 240 $\cos^2 \frac{1}{2}\,a + \sin^2 \frac{1}{2}\,a = 1$ (2)

whence by subtraction $2 \sin^2 \frac{1}{2}\,a = 1 - \cos a$

wherefore $\sin \frac{1}{2}\,a = (\frac{1}{2} - \frac{1}{2} \cos a)^{\frac{1}{2}}$

Adding next the equations 1 and 2, and proceeding in like manner, we obtain

$$\cos \tfrac{1}{2}\,a = (\tfrac{1}{2} + \tfrac{1}{2} \cos a)^{\frac{1}{2}}$$

Formulas for the tangent and cotangent of the sum and difference of two arcs.

244. By art. 45, we have $\tan(a+b)=\dfrac{\sin(a+b)}{\cos(a+b)}$;

whence by substitution $\tan(a+b)=\dfrac{\sin a\cos b+\sin b\cos a}{\cos a\cos b-\sin a\sin b}$;

dividing the numerator and denominator of the second member of this equation by $\cos a\cos b$, we have for the numerator

$$\frac{\sin a\cos b+\sin b\cos a}{\cos a\cos b}=\frac{\sin a}{\cos a}+\frac{\sin b}{\cos b}=\tan a+\tan b,$$

and for the denominator

$$\frac{\cos a\cos b-\sin a\sin b}{\cos a\cos b}=1-\frac{\sin a}{\cos a}\times\frac{\sin b}{\cos b}=1-\tan a\tan b;$$

whence by substitution and reduction, we obtain

$$\tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}. \qquad (1)$$

In like manner we obtain $\tan(a-b)=\dfrac{\tan a-\tan b}{1+\tan a\tan b}$. (2)

By art. 54, we have $\cot(a+b)=\dfrac{1}{\tan(a+b)}$; whence by substitution

$$\cot(a+b)=\frac{1-\tan a\tan b}{\tan a+\tan b}. \qquad (3)$$

So also

$$\cot(a-b)=\frac{1+\tan a\tan b}{\tan a-\tan b}. \qquad (4)$$

Formulas for the tangents and cotangents of multiple arcs.

245. Putting successively $b = a, 2a, 3a$, &c., in nos. 1 and 3 of the preceding article and substituting for factors of multiple arcs, we have

$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a}$$

$$\tan 3a = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}$$

$$\tan 4a = \frac{4\tan a - 4\tan^3 a}{1 - 6\tan^2 a + \tan^4 a}$$

$$\tan 5a = \&c.$$

The expressions for the cotangents will, it is easy to see, be merely the inverse of the preceding.

Formulas for the tangents and cotangents of half an arc.

246. By art. 45, we have $\tan \frac{1}{2} a = \frac{\sin \frac{1}{2} a}{\cos \frac{1}{2} a}$; whence by substitution art. 243,

$$\tan \tfrac{1}{2} a = \frac{(1 - \cos a)^{\frac{1}{2}}}{(1 + \cos a)^{\frac{1}{2}}}$$

multiplying numerator and denominator by $(1 + \cos a)^{\frac{1}{2}}$

$$\tan \tfrac{1}{2} a = \frac{(1 - \cos^2 a)^{\frac{1}{2}}}{1 + \cos a} = \frac{\sin a}{1 + \cos a}.$$

The expression for the cotangent $\frac{1}{2} a$, it is evident, will be the inverse of that for the tangent $\frac{1}{2} a$.

Formulas for secants and cosecants.

247. By art. 45 we have $\sec(a + b) = \frac{1}{\cos(a + b)}$; whence by substitution

$$\sec(a + b) = \frac{1}{\cos a \cos b - \sin a \sin b};$$

dividing both numerator and denominator by $\cos a \cos b$ and substituting we have

$$\sec(a+b) = \frac{\sec a \sec b}{1 - \tan a \tan b},$$

so also

$$\sec(a-b) = \frac{\sec a \sec b}{1 + \tan a \tan b}.$$

And by a process altogether similar we obtain

$$\operatorname{cosec}(a \pm b) = \frac{\sec a \sec b}{\tan a \pm \tan b}.$$

Making $b = a$ in the expressions for sec and cosec $(a+b)$, we have

$$\sec 2a = \frac{\sec^2 a}{1 - \tan^2 a}$$

$$\operatorname{cosec} 2a = \frac{\sec^2 a}{2 \tan a}.$$

General Formulas.

248. Adding and subtracting the last four of the fundamental equations, art. 240, we obtain

$$\sin(a+b) + \sin(a-b) = 2 \sin a \cos b \qquad (1)$$
$$\sin(a+b) - \sin(a-b) = 2 \sin b \cos a \qquad (2)$$
$$\cos(a+b) + \cos(a-b) = 2 \cos a \cos b \qquad (3)$$
$$\cos(a-b) - \cos(a+b) = 2 \sin a \sin b. \qquad (4)$$

If in these formulas we put $a = \frac{1}{2}(p+q)$, $b = \frac{1}{2}(p-q)$, we obtain

$$\sin p + \sin q = 2 \sin \tfrac{1}{2}(p+q) \cos \tfrac{1}{2}(p-q) \qquad (5)$$
$$\sin p - \sin q = 2 \sin \tfrac{1}{2}(p-q) \cos \tfrac{1}{2}(p+q) \qquad (6)$$
$$\cos p + \cos q = 2 \cos \tfrac{1}{2}(p+q) \cos \tfrac{1}{2}(p-q) \qquad (7)$$
$$\cos q - \cos p = 2 \sin \tfrac{1}{2}(p+q) \sin \tfrac{1}{2}(p-q) \qquad (8)$$

Making in these formulas $q = 0$, we have

$$\sin p = 2 \sin \tfrac{1}{2} p \cos \tfrac{1}{2} p \qquad (9)$$
$$1 + \cos p = 2 \cos^2 \tfrac{1}{2} p \qquad (10)$$
$$1 - \cos p = 2 \sin^2 \tfrac{1}{2} p. \qquad (11)$$

Dividing (9) by (10) and (11) successively we obtain

$$\frac{\sin p}{1+\cos p}=\frac{\sin \frac{1}{2} p}{\cos \frac{1}{2} p}=\tan \tfrac{1}{2} p \qquad (12)$$

$$\frac{\sin p}{1-\cos p}=\frac{\cos \frac{1}{2} p}{\sin \frac{1}{2} p}=\cot \tfrac{1}{2} p. \qquad (13)$$

Dividing next (5) by (7), we obtain

$$\frac{\sin p+\sin q}{\cos p+\cos q}=\frac{\sin \frac{1}{2}(p+q)}{\cos \frac{1}{2}(p+q)}=\tan \tfrac{1}{2}(p+q) \qquad (14)$$

From (5) and (8) we obtain in like manner

$$\frac{\sin p+\sin q}{\cos q-\cos p}=\cot \tfrac{1}{2}(p-q). \qquad (15)$$

And by operations altogether similar we obtain

$$\frac{\sin p-\sin q}{\cos p+\cos q}=\tan \tfrac{1}{2}(p-q) \qquad (16)$$

$$\frac{\sin p-\sin q}{\cos q-\cos p}=\cot \tfrac{1}{2}(p+q) \qquad (17)$$

$$\frac{\sin p+\sin q}{\sin (p+q)}=\frac{\cos \frac{1}{2}(p-q)}{\cos \frac{1}{2}(p+q)} \qquad (18)$$

$$\frac{\sin p-\sin q}{\sin (p+q)}=\frac{\sin \frac{1}{2}(p-q)}{\sin \frac{1}{2}(p+q)}. \qquad (19)$$

Recollecting that the cotangent is equal to 1 divided by the tangent, we obtain also

$$\frac{\sin p+\sin q}{\sin p-\sin q}=\frac{\tan \frac{1}{2}(p+q)}{\tan \frac{1}{2}(p-q)} \qquad (20)$$

$$\frac{\cos p+\cos q}{\cos q-\cos p}=\frac{\cot \frac{1}{2}(p+q)}{\tan \frac{1}{2}(p-q)}. \qquad (21)$$

249. From what has been done it will be perceived, that an indefinite number of trigonometrical formulas may be prepared, adapted to the purposes for which they may be required.

We have already developed, by aid of algebra, a rule by which to find the area of a triangle when the three sides are given. We close with the investigation of formulas for finding an angle when the sides are given.

Employing the same notation as in article 96, and resuming the expression for cos C there obtained, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

But this expression not being well adapted to calculation by logarithms, another is to be sought.

The formula $\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos 2a$ gives by substituting $\frac{1}{2}C$ for a, and deducing the value of cos C,

$$\cos C = 1 - 2\sin^2 \tfrac{1}{2} C;$$

comparing the two values of cos C, reducing to a common denominator, and recollecting that $c^2 - a^2 - b^2 + 2ab = c^2 - (a-b)^2 = (c + a - b)(c - a + b)$, we obtain

$$\sin^2 \tfrac{1}{2} C = \frac{\frac{1}{2}(c + a - b)\frac{1}{2}(c - a + b)}{ab};$$

hence, adding and subtracting b and a in the first and second factor of the numerator, representing $c + a + b$ by s and reducing, we have

$$\sin \tfrac{1}{2} C = \left(\frac{(\frac{1}{2}s - a)(\frac{1}{2}s - b)}{ab}\right)^{\frac{1}{2}}. \qquad (1)$$

Deducing next the value of cos C from the formula $\cos^2 a = \frac{1}{2} + \frac{1}{2}\cos 2a$, we obtain by a process altogether similar

$$\cos \tfrac{1}{2} C = \left(\frac{\frac{1}{2}s(\frac{1}{2}s - c)}{ab}\right)^{\frac{1}{2}} \qquad (2)$$

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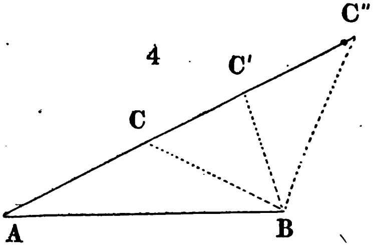

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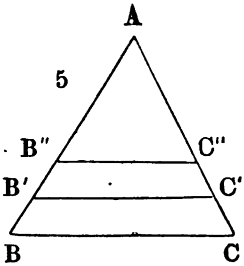

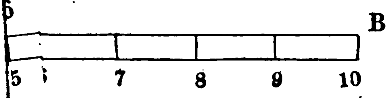

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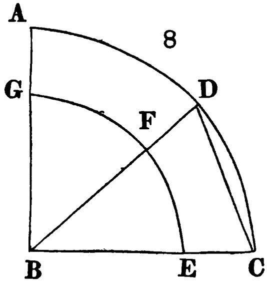

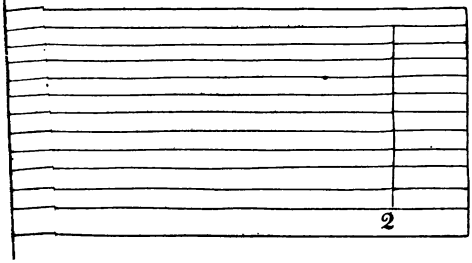

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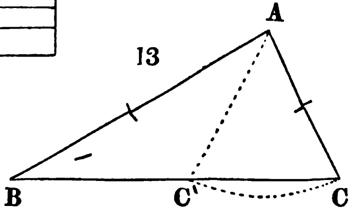

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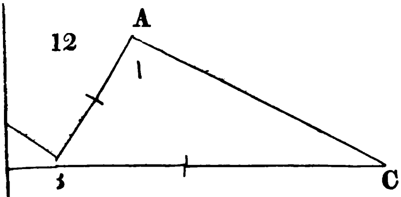

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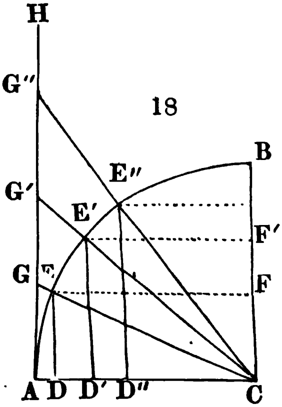

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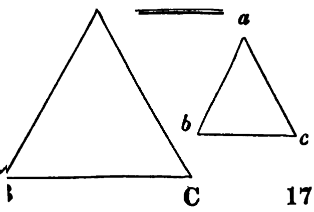

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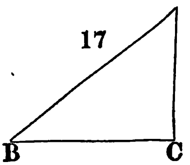

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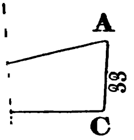

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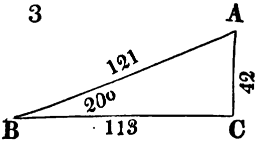

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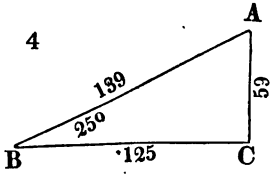

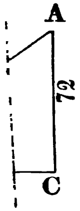

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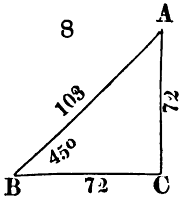

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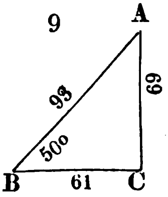

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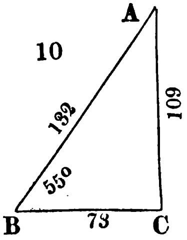

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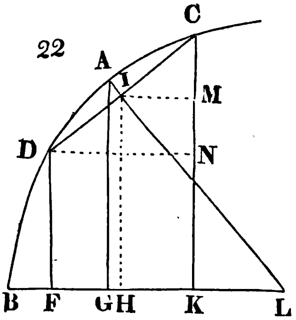

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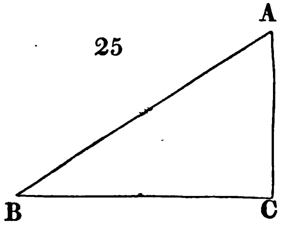

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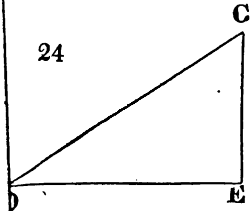

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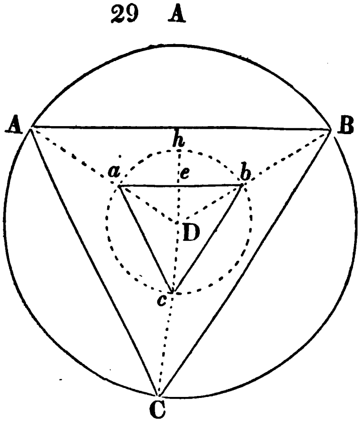

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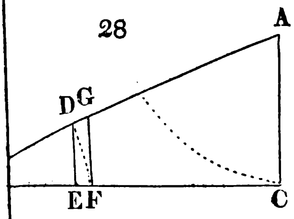

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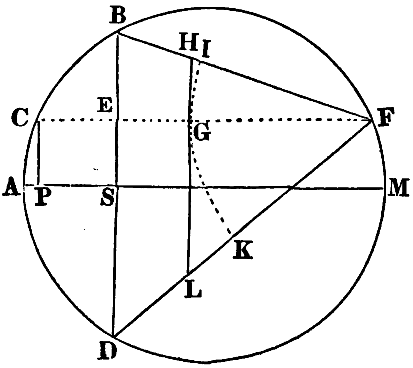

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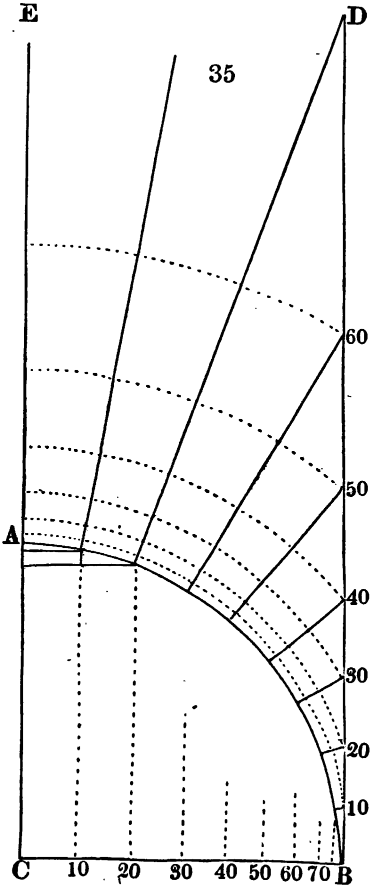

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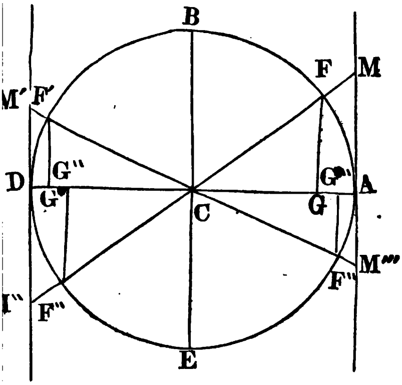

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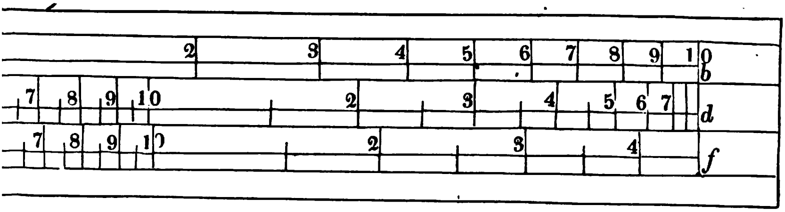

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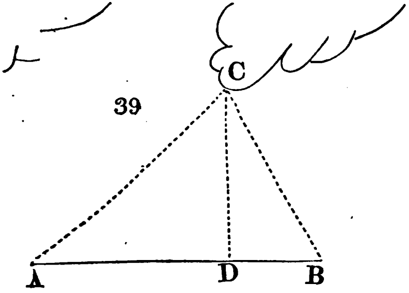

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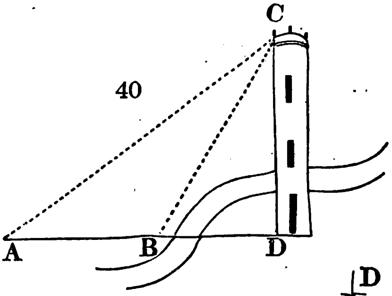

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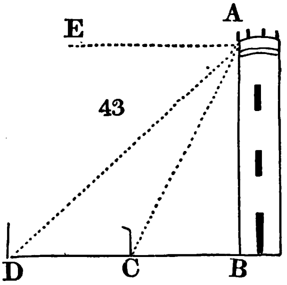

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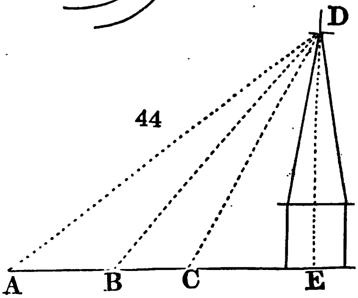

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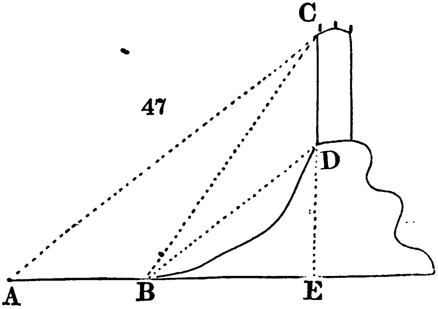

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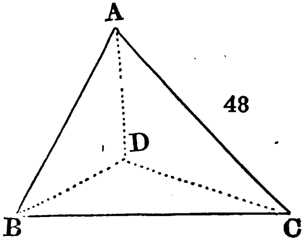

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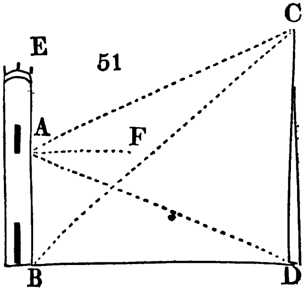

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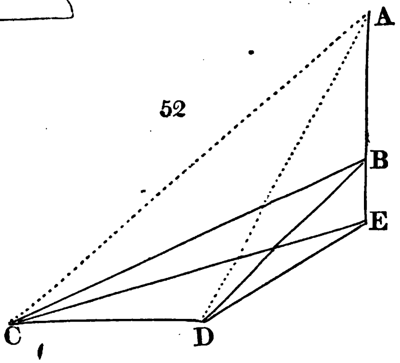

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